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CALCULATION OF FOUNDATION MESH SLABS ON AN ELASTIC LAYER

ABSTRACT

In this work, the authors have developed the procedure for calculation of mesh slabs on an elastic base modeled by an elastic homogeneous isotropic layer affected by the external load. The history of development of calculations of structures on an elastic basis demonstrates that, due to the scientific and technical progress, methods for calculation of aforementioned structures were improved and refined. This can be traced on various models of the elastic foundation that were used to simulate real soils in their natural occurrence or in an artificial base when setting up fundamentally new problems of structural analysis.

Variety of practical tasks results in ambiguous modelling of the elastic base. The authors refer to the works of Tarasevich A. N., Kozunova O. V. and Semenyuk S. D. that provide extensive systematic review of elastic base models for calculation of foundation beams, beam and foundation slabs, as well as for calculation of cross tapes for shallow foundations.

The relevance and timeliness of the proposed work is due to the fact that the issues of calculation of mesh slabs and the system of cross tapes on an elastic base have not yet been fully studied. The authors are familiar with the works of M. I. Gorbunov-Posadov, I. A. Simvulidi, G. Ya.

Popov, S. D. Semenyuk, S. N. Klepikov, where various approaches are used to conduct the researches in calculation of mesh slabs and spatial monolithic foundations as the system of cross tapes on an elastic base.

The procedure proposed is based on the Ritz variational method and the mixed method of structural mechanics using the Zhemochkin influence functions. To calculate the coefficients of canonical equations and the absolute terms for the mixed method of structural mechanics by way of the Zhemochkin method, the ratios of deflections with the normal restrained in the center of the slab are used in the calculation.

The numerical implementation of the new general-purpose approach is carried out, as an example, for the rectangular foundation slab with holes, symmetrically loaded by the uniformly distributed load, on the elastic uniform isotropic layer. Graphical results of calculations are given, describing the settlements of the foundation mesh slab and the distribution of contact stresses under the slab.

Keywords: foundation mesh slab, elastic base, elastic half-space, elastic uniform isotropic layer, Ritz variational method, Zhemochkin method, mixed method of structural mechanics, influence functions, settlements, contact stresses.

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РАСЧЕТ ФУНДАМЕНТНЫХ СЕТЧАТЫХ ПЛИТ НА УПРУГОМ СЛОЕ

АННОТАЦИЯ

В рассматриваемой работе авторами разработана методика расчета фундаментных сетчатых плит на упругом основании, моделируемом упругим однородным изотропным слоем, под действием внешней нагрузки. Из истории развития расчета конструкций на упругом основании видно, что методы их расчета совершенствовались и уточнялись по мере развития научно-технического прогресса. Это можно проследить на различных моделях упругого основания, которыми моделировались реальные грунты в естественном залегании или в искусственном основании при постановке принципиально новых задач расчета конструкций.

Разнообразие практических задач приводит к неоднозначному моделированию упругого основания. Авторы ссылаются на работы А. Н. Тарасевича, О. В. Козуновой и С. Д. Семенюка, в которых приведен обширный систематизированный обзор моделей упругого основания для расчета фундаментных балок, балочных и фундаментных плит, а также для расчета перекрестных лент фундаментов мелкого заложения.

Актуальность и своевременность предлагаемой работы в том, что вопросы расчета сетчатых плит и системы перекрестных лент на упругом основании до настоящего времени не исследованы в полной мере. Авторам известны *работы М. И. Горбунова-Посадова, И. А. Симвулиди, Г. Я. Попова, С. Д. Семенюка, С. Н. Клепикова, в которых различными подходами проведены исследования по расчету сетчатых плит и пространственных монолитных фундаментов, как системы перекрестных лент на упругом основании.*

Предлагаемая методика основана на вариационном методе Ритца и смешанном методе строительной механики с использованием функций влияния Жемочкина. Для определения коэффициентов канонических уравнений и свободных членов смешанного метода строительной механики через способ Жемочкина в расчете используются соотношения прогибов с защемленной в центре плиты нормалью.

Численная реализация нового универсального подхода выполнена на примере симметрично нагруженной равномерно-распределенной нагрузкой прямоугольной фундаментной плиты с отверстиями на упругом однородном изотропном слое. Приводятся графические результаты расчета для осадок фундаментной сетчатой плиты и распределения контактных напряжений под плитой.

Ключевые слова: фундаментная сетчатая плита, упругое основание, упругое полупространство, упругий однородный изотропный слой, вариационный метод Ритца, способ Жемочкина, смешанный метод строительной механики, функции влияния, осадки, контактные напряжения.

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INTRODUCTION

Calculation of structures on elastic bases is a branch of structural mechanics. The problem of calculation of these structures comprises the determination of reaction pressures (contact stresses) arising under the bed of foundation structures, and the determination of structure settlements. Also, determining the stress-strain state of a structure itself, lying on an elastic base [1, 2], is among the primary problems.

The history of development of the procedures for calculation of structures on an elastic base demonstrates that, due to the scientific and technical progress, methods for calculation of aforementioned structures were improved and refined [1-6]. This can be traced on various models of the elastic base that were used to simulate real soils in their natural occurrence or in an artificial base when setting up fundamentally new problems of structural analysis.

Variety of practical tasks results in ambiguous modelling of the elastic base. Choosing a computational model of the elastic base for various types of soils is especially difficult. The overview of models of the elastic base for calculation of solid-state foundation beams, beam slabs and foundation slabs is available in [7, 8]. The monograph by Semenyuk S.D., within the scope of the static calculation of cross tapes for shallow foundations, provides systematic description and classification of models of the elastic base [9] with further practical applications.

While the hypothesis of linear distribution of reactive stresses [1, 2, 5] was initially considered to be satisfactory, then, with the extensive construction of railway tracks and pontoon structures, the Fuss-Winkler-Zimmerman model gained momentum [10, 11]. However, it was also found to be inadequate for the real behaviour of cohesive soils. This has resulted in the elastic half-space model (for the problem formulation in spatial terms) and the generally-known Boussinesq's solution for this model [2, 12], and, for the flat-type problem formulation, the elastic half-space model with the Flamant solution [2, 8], respectively. Later, various modifications and combinations of the aforementioned models arose. The combination of the elastic half-space model and Winkler model is quite successfully appropriate for calculation of structures on layered non-homogeneous bases. These bases are used in arrangement of foundations on sandy cushions, and they are simulated by combined models [13, 14].

In mechanical terms, calculation of structures on the elastic base means solving the contact problem of contacting bodies [15]. These problems are reduced to solving the integral equations, with their solution being the function of the integral equation kernel and the shape of contacting bodies [16]. With simple forms of contacting bodies, the most difficult task is to determine the integral equation kernel also referred to as the Green function for contacting bodies [6, 15, 16]. The Green function is a function of displacements of points on the elastic base surface resulting from an impact of the concentrated unit force [6].

In engineering practice, solving each contact problem through the integral equations is reasonable due to extensive mathematical calculations. Therefore, Zhemochkin method [17] reducing the contact problem to the structural mechanics problem [20, 21] is used successfully for practical purposes.

The matters of calculation of foundation mesh slabs and the system of cross tapes on an elastic base have not been fully studied so far. Authors are familiar with the works of M. I. Gorbunov-Posadov [2], I. A. Simvulidi [18], G. Ya. Popov [19], S. D. Semenyuk [9], S. N. Klepikov [4], where various approaches are used to conduct researches in calculation of mesh slabs and spatial monolithic foundations as a system of cross tapes on an elastic base.

Below, the common approach is used for calculation of mesh slabs on a linearly deformable base that was proposed by the authors previously and tested numerically for the elastic half-space in the article [23]; also, it can be developed, taking physical nonlinearity of slab material into consideration [22]. This general-purpose approach is based on Zhemochkin method [17], making it possible to calculate, in unified terms, mesh slabs or the system of cross tapes, irrespective of their shape and stiffness, at various models of an elastic base for any vertical loads. The numerical implementation of the approach proposed is made by means of the example of a rectangular foundation slab with holes, on an elastic homogeneous isotropic base, loaded symmetrically by a uniformly distributed load.

The problem formulation and the linear calculation algorithm. The rectangular foundation slab is studied as a mesh slab on an elastic base, with the dimensions of LxB and with the uniform thickness of h, with rectangular holes (there dimensions are a) and b), loaded by the vertical load (Figure 1).

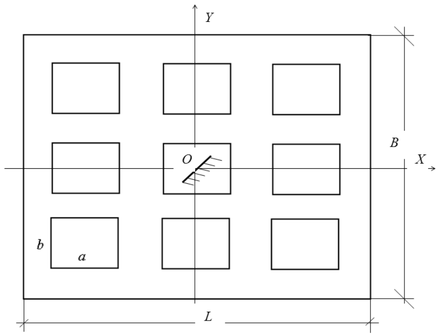

Figure 1. Rectangular foundation slab with holes, or mesh slab

The following hypotheses and assumptions are accepted:

- for a mesh slab or a foundation made of cross tapes, the thin slab bending hypotheses [15] are true;

- ties between the slab and the elastic base can be both in compression and in tension;

- shear stresses in the zone of contact between the slab and the base are disregarded;

- the length a and width b of holes in a slab meet the relation

$$2 < \left(\frac{Min(L,B)}{a}, \left(\frac{Min(L,B)}{b} \right) \right) < 6 .$$

These relations are usually applied in practice of construction of shallow foundations built as an in-situ concrete slab with holes, i.e. a mesh slab.

In the work proposed, the problem is formulated to calculate contact stresses under the mesh slab bed, the mesh slab settlements and internal forces in a slab resulting from vertical load.

Solving the problem. We use Zhemochkin method [17] to solve the problem. We divide the slab into equal rectangular fragments and put a tie in the centre of each fragment to implement contact between the slab and the elastic base. It should be noted that the tape width must be divided into two or more fragments to take the effect of torques into consideration. We assume that the force in a tie results in a uniform distribution of contact stresses within each fragment.

We compose the system of linear algebraic equations (SLE) for the mixed method of structural mechanics

$$\begin{cases} \delta_{1,1}X_1 + \ldots + \delta_{1,n}X_n + u_0 - \phi_x y_1 - \phi_y x_1 + \Delta_{1,F} = 0 \\ \ldots \\ \delta_{n,1}X_1 + \ldots + \delta_{n,n}X_n + u_0 + \phi_x y_n + \phi_y x_n + \Delta_{n,F} = 0 \\ -\sum_{i=1}^{n} X_i + R = 0 \\ \sum_{i=1}^{n} X_i y_i + M_{xR} = 0 \\ \sum_{i=1}^{n} X_i x_i + M_{yR} = 0, \end{cases} \qquad (1)$$

where $\delta_{i,j}$ is a mutual displacement of the cross-cut tie with the number i resulting from the impact of unit forces applied in the j^{th} Zhemochkin tie used to implement the contact between the slab and the elastic base. It is a sum of two items [17]: X_i is an unknown force in the i^{th} Zhemochkin tie; u_0, φ_x, φ_y are unknown vertical and angular displacements in the restraint inserted in the centre of the slab; x_i, y_i are the coordinates of the centre of the rectangular Zhemochkin fragment with the number i; $\Delta_{i,F}$ is a displacement of the point i in the mesh slab with the restrained normal due to the impact of the external load F; R, M_{xR}, M_{yR} are the resultant of the external load and the torque of the resultant about the axis X and Y respectively; n is the number of Zhemochkin fragments.

The coefficients at unknown values $\delta_{i,j}$ in the SLE for the elastic half-space with the modulus of elasticity E_0 and the Poisson ratio ν_0 are calculated using the equation [17]

$$\delta_{i,k} = \frac{1-\nu_0^2}{\pi E_0 \Delta x} F_{i,k} + \frac{L^2}{D} W_{i,k}, \tag{2}$$

where $W_{i,k}$ is a vertical displacement of the middle of the ith fragment in the mesh slab with the restrained normal due to the impact of the unit force applied in the point k of the mesh slab; L is the slab size along the direction of the axis OX; Δx is the size of the Zhemochkin fragment along the axis OX; $F_{i,k}$ is the dimensionless function for determination of the vertical displacement of the point i on the surface of the elastic base due to the impact of the unit force distributed over the fragment with the number k; ν_0, E_0 are the Poisson ratio and the modulus of elasticity of the elastic base.

The dimensionless function $F_{i,k}$ in the equation (2) are calculated using the equations from the monograph [14], namely

$$F_{i,i} = 2\frac{c}{b}\left[\ell n\frac{b}{c} + \frac{b}{c}\ell n\left(\frac{c}{b} + \sqrt{\frac{c^2}{b^2}+1}\right) + \ell n\left(1 + \sqrt{\frac{c^2}{b^2}+1}\right)\right];$$

$$F_{i,k} = \frac{1}{\left|x_i - x_k\right|}. \tag{3}$$

For an elastic homogeneous isotropic layer pivotally connected with the non-deformable base, vertical displacements of the elastic layer surface due to the concentrated force P are calculated using the equation (2.13) from the monograph [6]

$$W(R) = \frac{P(1-\nu_0^2)}{\pi E_0}\left[\frac{1}{R} + \frac{1}{h}\sum_{h=0}^{\infty} a_n \frac{\Gamma(n+1)}{\left(4+\frac{R^2}{h^2}\right)^{\frac{n+1}{2}}} P_n\left(\frac{2h}{\sqrt{R^2+4h^2}}\right.\right. \tag{4}$$

where $R=\sqrt{x^2+y^2}$ is the radius vector of moving points with the coordinates (x, y) on the analysed surface of the elastic layer;

h is the elastic layer thickness, m;

$\Gamma\,(n+1)$ is the gamma function [17];

$P_n\left(\frac{2h}{\sqrt{R^2+4h^2}}\right)$ is the Legendre polynomial [17];

a_n are indefinite coefficients for expansion in series.

The following values were calculated in the monograph [6]: $a_0 = -1$; $a_1 = -3/2$; $a_2 = -1$; $a_3 = -1/3$; $a_4 = 1/18$;...

After integration (4) over the area of the rectangular fragment with the sizes of $\Delta x \cdot \Delta y$, we obtain the equations for calculation of displacements of the centre of Zhemochkin fragment with the number i due to the impact of the concentrated force equal to 1, applied in the centre of the fragment with the number k.

The first addend in the equation (4) determines the function of vertical displacements for the elastic homogeneous isotropic half-space (Boussinesq's solution), it is integrated precisely (it is singular); other addends are not singular and not integrated. For practical calculations in the equation (4), the series [6] can be limited to five terms only.

The following equation for displacement of the point M (x_i, y_i) was also derived in the monograph [6]

$$W(x_i,y_i)=\frac{1-\nu_0^2}{\pi E_0 \Delta x}\left[\begin{array}{l}\frac{y_i-d}{\Delta y}\ln\frac{x_i-b+\sqrt{(x_i-b)^2+(y_i-d)^2}}{x_i-a+\sqrt{(x_i-a)^2+(y_i-d)^2}}+ \\ +\frac{y_i-c}{\Delta y}\ln\frac{x_i-a+\sqrt{(x_i-a)^2+(y_i-c)^2}}{x_i-b+\sqrt{(x_i-b)^2+(y_i-c)^2}}+ \\ +\frac{x_i-b}{\Delta y}\ln\frac{y_i-d+\sqrt{(x_i-b)^2+(y_i-d)^2}}{y_i-c+\sqrt{(x_i-b)^2+(y_i-c)^2}}+ \\ +\frac{x_i-a}{\Delta y}\ln\frac{y_i-c+\sqrt{(x_i-a)^2+(y_i-c)^2}}{y_i-d+\sqrt{(x_i-a)^2+(y_i-d)^2}}\end{array}\right]+ \qquad (5)$$

$$+\frac{1-\nu_0^2}{\pi E_0 h}\sum_{h=0}^{\infty} a_n \frac{\Gamma(n+1)}{\left(4+\frac{(x_i-x_k)^2+(y_i-y_k)^2}{h^2}\right)^{\frac{n+1}{2}}} P_n\left(\frac{2h}{\sqrt{(x_i-x_k)^2+(y_i-y_k)^2+4h^2}}\right).$$

Let's write the equation (5) in terms of the dimensionless function $F_{i,k}$, namely

$$W(x_i, y_i) = \frac{1-\nu_0^2}{\pi E_0 \Delta x} \cdot F_{ik}, \tag{6}$$

where

$$F_{ik} = \left[\begin{array}{l} \frac{y_i - d}{\Delta y} \ln \frac{x_i - b + \sqrt{(x_i - b)^2 + (y_i - d)^2}}{x_i - a + \sqrt{(x_i - a)^2 + (y_i - d)^2}} + \\ + \frac{y_i - c}{\Delta y} \ln \frac{x_i - a + \sqrt{(x_i - a)^2 + (y_i - c)^2}}{x_i - b + \sqrt{(x_i - b)^2 + (y_i - c)^2}} + \\ + \frac{x_i - b}{\Delta y} \ln \frac{y_i - d + \sqrt{(x_i - b)^2 + (y_i - d)^2}}{y_i - c + \sqrt{(x_i - b)^2 + (y_i - c)^2}} + \\ + \frac{x_i - a}{\Delta y} \ln \frac{y_i - c + \sqrt{(x_i - a)^2 + (y_i - c)^2}}{y_i - d + \sqrt{(x_i - a)^2 + (y_i - d)^2}} \end{array}\right] + \tag{7}$$

$$+\frac{\Delta x}{h}\sum_{h=0}^{\infty} a_n \frac{\Gamma(n+1)}{\left(4+\frac{(x_i-x_k)^2+(y_i-y_k)^2}{h^2}\right)^{\frac{n+1}{2}}} P_n\left(\frac{2h}{\sqrt{(x_i-x_k)^2+(y_i-y_k)^2+4h^2}}\right).$$

The greatest difficulties arise from determination of deflections of a mesh slab with the normal restrained in its center. Supposing that the slab is solid, we could use the solution given in the monograph [9].

Therefore, in the work under consideration, to calculate the deflections of a mesh slab with the normal restrained in its center, the Ritz method was applied where, as the basis functions, the first five partial Clebsch solutions [6] were accepted, meeting the boundary conditions in terms of displacements in the restraint. Therefore, the following deflection function was accepted for calculation of the deformation energy [6]

$$W(x,y)=A_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+A_1\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+A_2\frac{xy}{ab}+A_3\frac{x}{a}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+A_4\frac{y}{b}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right), \quad (8)$$

where A_i are unknown coefficients calculated according to the Ritz method [21], subject to the condition of the minimum of the potential energy of the slab with the reatrained normal and the concentrated force applied to it.

Then, the system (1) was prepared and solved, and calculated forces in Zhemochkin ties were used to calculate the mesh slab settlements; they, in turn, were used to calculate torques and transverse forces in slab sections according to known equations [15] and to calculate contact stresses in the zone of contact of the slab with the elastic base.

Calculation results. Let's calculate the square reinforced-concrete slab with the plan dimensions 13 m x 13 m, with four square holes, $a = b = 5$ m. The flexural stiffness of the slab is $D = 5000$ kN·m. The slab is placed on the elastic half-space and the elastic homogeneous isotropic layer, pivotally connected with the non-deformable base with constant parameters of elasticity, $\nu_0 = 0.35$ and $E_0 = 20$ MPa, affected by the uniformly distributed load, 10 kPa.

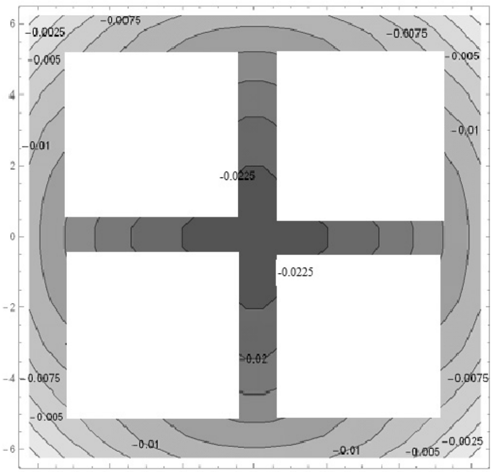

Figure 2. Equal displacement curves (m) for the mesh slab on the elastic half-space

The slab was subdivided into 156 equal rectangular Zhemochkin fragments and in two fragments along the tape width.

See Figures 2, 3 for equal displacement (settlement) curves for the mesh slab, and Figures 4, 5 for equal contact stress curves for the mesh slab, with different models of the elastic base (see Figures 2, 4 for the elastic half-space, and Figures 3, 5, for the elastic layer).

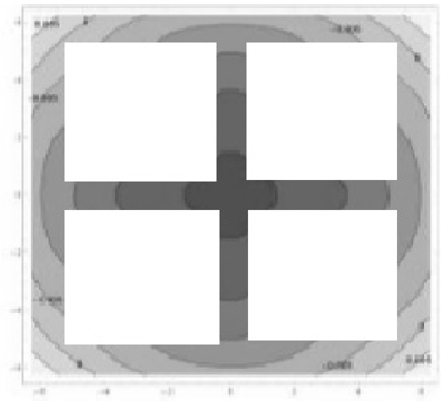

Figure 3. Equal displacement curves (m) for the mesh slab on the elastic isotropic layer

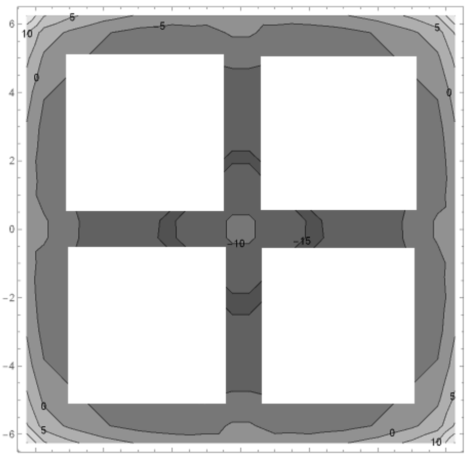

Figure 4. Equal contact stress curves (kPa) for the mesh slab on the elastic half-space

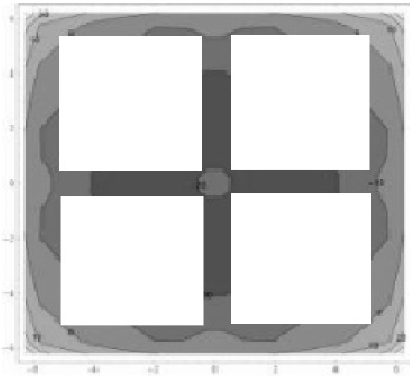

Figure 5. Equal contact stress curves (kPa) for the mesh slab on the elastic isotropic layer

SUMMARY AND CONCLUSIONS

1. The article offers non-sophisticated but general-purpose approach, based on the Zhemochkin method, for calculation of a mesh slab on an elastic base affected by the vertical load. The example described provides the solution both for the elastic half-space and the elastic isotropic layer. This result confirms the fact that the dimensionless function $F_{i,k}$ in the equation (2) is a characteristic of the elastic base model, and its variation in accordance with [17] allows calculations of mesh slabs with various elastic base models.

2. To calculate the mesh slab according to Winkler model, the equation for the influence function (2) is assumed to be $F_{i,\hat{e}} = 0, i \neq k$ and $F_{i,i} = \frac{1,}{K\,\Delta x \Delta y}$, where K is the Winkler bedding coefficient for the base.

3. Using the results given in authors' work [22], the approach offered can be generalized to the calculation of the mesh slab, with its material's physical non-linearity taken into consideration.

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