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MILD SOLUTION OF THE CAUCHY PROBLEM FOR A SEMILINEAR NONSTRICTLY HYPERBOLIC EQUATION ON A HALF-PLANE IN THE CASE OF A SINGLE CHARACTERISTIC

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Abstract. For a semilinear nonstrictly hyperbolic equation on a half-plane in the case of a single characteristic given in the upper half-plane, we consider the Cauchy problem, for which we study issues related to the mild solution.
Key words: Cauchy problem, nonstrictly hyperbolic equation, semilinear equation, mild solution.

СЛАБОЕ РЕШЕНИЕ ЗАДАЧИ КОШИ ДЛЯ ПОЛУЛИНЕЙНОГО НЕСТРОГО ГИПЕРБОЛИЧЕСКОГО УРАВНЕНИЯ НА ПОЛУПЛОСКОСТИ В СЛУЧАЕ ОДНОЙ ХАРАКТЕРИСТИКИ

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Аннотация. Для полулинейного нестрого гиперболического уравнения на полуплоскости в случае единственной характеристики, заданной в верхней полуплоскости, рассматривается задача Коши, для которой изучаются вопросы, связанные со слабым решением.
Ключевые слова: задача Коши, нестрого гиперболическое уравнение, полулинейное уравнение, слабое решение.

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Statement of the problem. In the domain $Q = (0,\infty)\times\mathbb{R}$, consider the m^{th}-order nonlinear differential equation

$$(\partial_t - a\partial_x + b)^m u(t,x) = f\big(t,x,u(t,x)\big), \quad (1)$$

where a and b are given real numbers, satisfying the condition $a \neq 0$ (it means that the line $t = 0$ is not the characteristic of Eq. (1)), and f is a function given on the set $\bar{Q}\times\mathbb{R}$.

Equation (1) is equipped with the initial condition

$$\partial_t^i u(0,x) = \varphi_i(x),\ i = 0,\dots,m-1,\ x\in\mathbb{R}, \quad (2)$$

where φ_i are functions given on the real axis.

Eq. (1) describes a wavefield resulting from the superposition of m waves traveling in one direction with equal velocity. When $m = 1$, Eq. (1) is called the one-dimensional transport equation. Equations of the kind (1) appear in many physical phenomena where discontinuous or singular entities are involved, for instance, in the wave propagation in a layered medium [1]. Eq. (1) is also used for the modeling k-out-of-n systems [2] and can have some applications in classical field theory.

The existence and uniqueness of classical solutions of the problem (1), (2) were studied in our preprint [3].

Reduction to the Cauchy problem for an ordinary differential equation. Making the linear nondegenerate change of independent variables $\tau = t,\ \xi = x + at$, and denoting $u(t,x) = v(\tau,\xi)$, we obtain the new differential equation

$$(\partial_\tau + b)^m v(\tau,\xi) = F\big(\tau,\xi,v(\tau,\xi)\big), \quad (3)$$

where $F(\tau,\xi,\mathrm{v}) = f(\tau,\xi - a\tau, v)$. The initial conditions for the function v can be computed using the Faá di Bruno′s formula or the chain rule, and they have the form

$$v(0,\xi) = \widetilde{\varphi}_0(\xi) = \varphi_0(\xi),\ \partial_\tau^i v(0,\xi) = \widetilde{\varphi}_i(\xi) = \\ = \textstyle\sum_{j=0}^{i}\binom{i}{j}(-a)^{i-j}D^{i-j}\varphi_j(\xi),\ i = 1,\dots,m-1. \quad (4)$$

Now Eq. (3) with the conditions (4) can be considered as the Cauchy problem for an ordinary differential equation with the parameter ξ, i. e.,

$$(D+b)^m v_\xi(\tau) = F\left(\tau,\xi,v_\xi(\tau)\right), \quad (5)$$

$$D^i v_\xi(0) = \widetilde{\varphi}_{\mathrm{i}}(\xi),\ i = 0,\dots,m-1, \quad (6)$$

We can say that the problems (1), (2) and (5), (6) are the same in the sense that the first is written in Eulerian coordinates (t,x) and the second in Lagrangian coordinates (τ,ξ).

To simplify Eq. (5), we use the following ansatz

$$v_\xi(\tau) = \mathrm{w}_\xi(\tau)\exp(-b\tau). \quad (7)$$

Substituting (7) into (5), we obtain the equation

$$D^m w_\xi(\tau) = \Phi\left(\tau,\xi,w_\xi(\tau)\right), \quad (8)$$

where

$$\Phi(\tau,\xi,w) = F(\tau,\xi,w\exp(-b\tau))\exp(b\tau).$$

The Cauchy conditions have the form

$$w_\xi(0) = \psi_0(\xi) = \exp(b\tau)\,\tilde{\varphi}_0(\xi),$$
$$D^i w_\xi(0) = \psi_i(\xi) = \qquad (9)$$
$$= \sum_{j=0}^{i} \binom{i}{j} b^j \tilde{\varphi}_{i-j}(\xi),\ i = 1, \dots, m-1.$$

Now we can use the theory of generalized solutions for ordinary differential equations to construct generalized solutions of the Cauchy problem (1), (2).

Note that the smoothness of the "new" initial conditions is not worse than the "old" ones in the sense that $\varphi_i \in C^{n-i}(\Omega)$ if and only if $\tilde{\varphi}_i \in C^{n-i}(\Omega)$ and $\tilde{\varphi}_i \in C^{n-i}(\Omega)$ if and only if $\psi_i \in C^{n-i}(\Omega)$, where $\Omega \subseteq \mathbb{R}$, $n \geq m-1$, and $i = 0, \dots, m-1$.

Mild solution. The classical and mild solutions of the problem (8), (9) can be represented as [4]

$$w_\xi(\tau) = \sum_{i=0}^{m-1} \frac{\psi_i(\xi)\tau^i}{i!} + \frac{1}{(m-1)!} \times$$
$$\times \int_0^\tau \Phi\left(\tau_1, \xi, w_\xi(\tau_1)\right)(\tau - \tau_1)^{m-1} d\tau_1.$$

Returning to the original variables, we get

$$u(t,x) = \sum_{i=0}^{m-1} \frac{\psi_i(x+at)\exp(-bt)\,t^i}{i!} +$$
$$+ \frac{1}{(m-1)!} \int_0^t \left[\exp\left(-b(t-\tau)\right)(t-\tau)^{m-1} \times \right. \qquad (10)$$
$$\left. \times f(\tau, x + a(t-\tau), u(\tau, x + a(t-\tau))\right] d\tau.$$

We can use Eq. (10) to define a mild solution of the Cauchy problem (1), (2).

Definition 1. *The function* u *is a mild solution of the Cauchy problem* (1), (2) *if it is a solution of Eq.* (10).

Theorem 1. *Let the conditions* $f \in C(\bar{Q} \times R)$, *and* $\varphi_i \in C^{m-i-1}(\mathbb{R})$, $i = 0, \dots, m-1$, *be satisfied, and let the function* f *satisfy the Lipschitz condition*

$$|f(t,x,z_1) - f(t,x,z_2)| \leq K(t,x)|z_1 - z_2|,$$

where $K \in C(\bar{Q} \times \mathbb{R})$. *The Cauchy problem* (1), (2) *has a unique mild solution in the class* $C(\bar{Q})$.

The **proof** of the theorem is carried out using the Leray–Schauder fixed point theorem.

Note that, in contrast to strictly hyperbolic equations [5], here we have to increase the smoothness of the initial data by $m-1$ times to construct a weak solution because the functions ψ_i, $i = 0, \dots, m-1$, must be continuous and defined everywhere. It is because the characteristic has a multiplicity m. Any characteristic of multiplicity k entails increasing the smoothness of the initial data by $k-1$ times to construct a well-defined solution [6].

Remark 1. *In Theorem 1, the smoothness conditions* "$\varphi_i \in C^{m-i-1}(\mathbb{R})$, $i = 0, \dots, m-1$" *can be weakened to «the functions* φ_i, $i = 0, \dots, m-1$, *have all derivatives up to order* $m-i-1$, *which are defined everywhere on the set R and are piecewise continuous». But the solution will no longer be continuous on the set* $[0,\infty) \times \mathbb{R}$. *Instead, it will be discontinuous on some characteristics* $x + at = \text{const}$.

Further weakening of the smoothness conditions for the initial data to piecewise smooth functions or to functions belonging to the Sobolev spaces can lead to difficulties in defining the functions ψ_i, $i = 0, \dots, m-1$, since a discontinuous function has no derivative, even in a weak sense.

Theorem 2. *Let the conditions* $f \in C(\bar{Q} \times \mathbb{R})$ *and* $\partial_u f(\cdot_1, \cdot_2, u = \cdot_3) \in C(\bar{Q} \times \mathbb{R})$ *be satisfied. The Cauchy problem* (1), (2) *has at most one mild solution defined on the set* $\bar{Q}$ *in the class of measurable functions, which are bounded on every compact subset of* $\bar{Q}$.

The **proof** of the Theorem 2 is carried out by contradiction using the mean value theorem and the Grönwall inequality.

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