Белорусский национальный технический университет

Факультет Информационных Технологий и Робототехники

Кафедра «Высшая математика»

Заведующий кафедрой

Щукин М.В._________ 2023г.

Декан факультета

Авсиевич А.М.________ 2023г.

MATHEMATICS 1

Электронный учебно-методический комплекс
для учащихся специальности
6-05-0713-05 «Робототехнические системы»

Составители: Г.К. Воронович, М.Н. Королева, В.И. Юринок, М.В. Щукин, Н.И. Чепелев, А.Н. Чепелев, М.А. Хотомцева

Минск
БНТУ
2023

При подготовке ЭУМК использовались материалы ЭУМК «Математика. Часть 1», изданного в БНТУ в 2021 г., составители: Лебедева Г.И., Герасимова Е.А., Раевская Л.А., Сагарда Е.В., Юринок В.И., Яцкевич Т.С.

CONTENTS

1 ПЕРЕЧЕНЬ МАТЕРИАЛОВ

Электронный учебно-методический комплекс по учебной дисциплине «Mathematics1» состоит из следующих разделов:

– кратких теоретических материалов по курсу математики первого семестра обучения;

– материалов для проведения практических занятий по учебной дисциплине;

– материалов для текущей и итоговой аттестации;

– вспомогательных материалов.

Теоретический раздел ЭУМК содержит материалы для теоретического изучения учебной дисциплины в объеме, установленном учебным планом по специальности.

Практический раздел ЭУМК содержит материалы для проведения практических занятий в аудитории и заданий для самостоятельной работы.

Раздел контроля знаний ЭУМК содержит материалы текущей и итоговой аттестации, позволяющие определить соответствие результатов учебной деятельности обучающихся требованиям образовательных стандартов высшего образования и учебно-программной документации, и представлен типовыми расчетами по темам учебной дисциплины и тестами. В разделе тестов приведен пример их решения и размещены ответы ко всем тестам.

Вспомогательный раздел ЭУМК содержит программу дисциплины, экзаменационные вопросы, перечень учебно-методических пособий, рекомендуемых к использованию в образовательном процессе.

2 ПОЯСНИТЕЛЬНАЯ ЗАПИСКА

ЭУМК предназначен для изучения дисциплины «Mathematics» для иностранных студентов, владеющих английским языком. Он содержит набор методических материалов по этой дисциплине.

ЭУМК состоит из четырех частей.

Теоретический раздел содержит набор методических материалов по этому предмету: рекомендаций студенту для работы с дисциплиной, кратких теоретических материалов, посвященных изложению в наглядном виде основных определений, свойств, формул и теорем, сопровождающихся подробными примерами.

Практический раздел содержит практикум по дисциплине, состоящий из материалов для проведения аудиторных занятий по математике. Каждое занятие содержит задачи для домашней работы с ответами.

Раздел контроля знаний содержит типовые расчеты, тесты для организации текущего контроля знаний студентов и контрольные работы (30 вариантов) для студентов заочного отделения.

Вспомогательный раздел содержит программу дисциплины, перечень экзаменационных вопросов, список рекомендуемой.

Конспект лекций в ЭУМК представляет собой гипертекстовый pdf-документ, предоставляющий возможность навигации по содержанию документа. Все задачи в практикуме снабжены ответами, которые могут быть использованы для самоконтроля. В конце каждого раздела практикума предложены 30 вариантов типовых расчетов, предназначенных для самостоятельного выполнения. Тестовые задания при текущем контроле могут быть выполнены как в аудитории, так и в компьютерной системе тестирования.

3 ОБЩИЕ РЕКОМЕНДАЦИИ СТУДЕНТУ ПО РАБОТЕ НАД ДИСЦИПЛИНОЙ «МАТЕМАТИКА»

Основными формами обучения студента являются лекции, практические занятия и самостоятельная работа над учебным материалом, которая состоит из следующих этапов:

- изучения теоретического материала по учебникам, учебным пособиям, конспектам лекций и т.д.;
- решения задач и упражнений;
- выполнения контрольных работ.

Рассмотрим некоторые из них.

Работа с учебником

1. Изучая материал по учебнику, к следующему вопросу следует переходить только после правильного понимания предыдущего, производя самостоятельно на бумаге все вычисления (в том числе и те, которые ради краткости опущены в учебнике) и выполняя имеющиеся в учебнике чертежи.
2. Особое внимание следует обращать на определение основных понятий. Студент должен подробно разбирать примеры, которые поясняют такие определения, и уметь строить аналогичные примеры самостоятельно.
3. Необходимо помнить, что каждая теорема состоит из предположений и утверждения. Все предположения должны обязательно использоваться в доказательстве. Нужно точно представлять то, в каком месте доказательства использовано каждое предположение теоремы. Полезно составлять схему доказательств теорем. Правильному пониманию многих теорем помогает разбор примеров математических объектов, обладающих и не обладающих свойствами, указанными в предположениях и утверждениях теорем.
4. При изучении материала по учебнику полезно вести конспект, в который рекомендуется вписывать определения, формулировки теорем, формулы, уравнения и т.д. На полях следует отмечать вопросы, выделенные студентом для получения письменной или устной консультации преподавателя.

5. Процесс письменного оформления работы студента с учебником имеет исключительно важное значение. Записи в конспекте должны быть сделаны чисто, аккуратно и расположены в определенном порядке. Хорошее внешнее оформление конспекта по изученному материалу приучает студента к необходимому в работе порядку и позволяет ему избежать многочисленных ошибок, которые происходят из-за небрежных, беспорядочных записей.

6. Выводы, полученные в виде формул, рекомендуется в конспекте подчеркивать или обводить рамкой (желательно чернилами другого цвета), чтобы при перечитывании конспекта они выделялись и лучше запоминались. Опыт показывает, что многим студентам помогает в работе составление листа, содержащего важнейшие и наиболее часто употребляемые формулы курса. Такой лист не только помогает запомнить формулы, но и может служить постоянным справочником для студента.

Список рекомендованной литературы приведен в конце методических указаний.

Решение задач

1. Чтение учебника должно сопровождаться решением задач, для чего рекомендуется завести специальную тетрадь.

2. При решении задач нужно обосновать каждый этап решения, исходя из теоретических положений курса. Если студент видит несколько путей решения, то он должен сравнить их и выбрать самый лучший. До начала вычислений полезно составить краткий план решения.

3. Решения задач и примеров следует излагать подробно, вычисления располагать в строгом порядке, отделяя вспомогательные вычисления от основных. Чертежи можно выполнять от руки, но аккуратно и в соответствии с данными условиями. Если чертеж требует особо тщательного выполнения (например, при графической проверке решения, полученного путем вычислений), то следует пользоваться линейкой, транспортиром, лекалом и указывать масштаб.

4. Решение каждой задачи должно доводиться до ответа, требуемого условием, и, по возможности, в общем виде с выводом формулы. Затем в полученную формулу подставляют числовые значения (если они даны). В промежуточных вычислениях не следует вводить приближенные значения корней, числа π и т.д.

5. Полученный ответ следует проверять способами, вытекающими из существа данной задачи. Если, например, решалась задача с конкретным физическим или геометрическим содержанием, то полезно, прежде всего, проверить размерность полученного ответа. Полезно также, если возможно, решить задачу несколькими способами и сравнить полученные результаты.

6. Решение задач определенного типа нужно продолжать до приобретения твердых навыков в их решении.

Самопроверка

1. После изучения определенной темы по учебнику и решения достаточного количества соответствующих задач студенту рекомендуется воспроизвести по памяти определения, выводы формул, формулировки и доказательства теорем.

В случае необходимости надо еще раз внимательно разобраться в материале учебника, решить ряд задач.

2. Иногда недостаточность усвоения того или иного вопроса выясняется только при изучении дальнейшего материала. В этом случае надо вернуться назад и повторить плохо усвоенный раздел.

3. Важным критерием усвоения теории является умение решать задачи на пройденный материал. Однако здесь следует предостеречь студента от весьма распространенной ошибки, заключающейся в том, что благополучное решение задач воспринимается им как признак усвоения теории. Часто правильное решение задачи получается в результате применения механически заученных формул, без понимания существа дела. Можно сказать, что умение решать задачи является необходимым, но не достаточным условием хорошего знания теории.

Консультации

1. Если в процессе работы над изучением теоретического материала или при решении задач у студента возникают вопросы, разрешить которые самостоятельно не удается (неясность терминов, формулировок теорем, отдельных задач и др.), то он может обратиться к преподавателю для получения от него письменной или устной консультации.

2. В своих запросах студент должен точно указать, в чем он испытывает затруднение. Если он не разобрался в теоретических объяснениях, или в доказательстве теоремы, или в выводе формулы по учебнику, то нужно указать, какой это учебник, год его издания и страницу, где рассмотрен затрудняющий его вопрос, и что именно его затрудняет. Если студент испытывает затруднение при решении задачи, то следует указать характер этого затруднения, привести предполагаемый план решения.

3. За консультацией следует обращаться и при сомнении в правильности ответов на вопросы для самопроверки.

Краткие советы-рекомендации студентам по организации своей учебно-профессиональной деятельности в ходе вузовского обучения

Вступительные экзамены позади, и теперь Вы можете гордо заявить: «Я – студент БНТУ!»

Казалось бы, можно вздохнуть с облегчением – страхи и волнения позади, а впереди – новая и интересная студенческая жизнь. Но расслабляться еще рано – именно первый курс профессионального обучения является наиболее трудным.

Будьте готовы к тому, что обучение в профессиональном учебном заведении существенно отличается от обучения в школе: учебная нагрузка больше и предметы сложнее; от студента требуется максимум самостоятельности и ответственности в изучении дисциплин; для успешного обучения необходимы такие качества как организованность и развитый самоконтроль.

Ваша успеваемость, а, следовательно, и уровень Вашей подготовки как будущего специалиста во многом зависит от первых месяцев пребывания в университете. Сумели адаптироваться – успеваете, не сумели – отстаете.

Наиболее общими причинами отставания многих студентов является неорганизованность, неумение распределять рабочее время для самостоятельной подготовки, недостаточная подготовленность в средней школе, неумение быстро приспосабливаться к новым формам и методам вузовского обучения, увлечение другими видами деятельности, отсутствие регулярного контроля над ходом учебы.

Первокурснику предстоит:

Осознать себя в новом качестве («Я – студент»).

Начинается все с того, что молодой человек ощущает что, поступив, сделал что-то очень важное, поднялся на новую жизненную ступень. Но, попав в студенческое сообщество, понимает, что он ничем не выделяется – все одногруппники в одинаковом положении. Весь авторитет, заработанный в школе, мало кого интересует, и предстоит заявлять о себе заново, прежде чем тебя начнут серьезно воспринимать. Но в этой ситуации есть и положительный момент – для учеников, чьи успехи в школе были не блестящи, это прекрасная возможность начать все с чистого листа и проявить себя с лучшей стороны.

Влиться в студенческий коллектив.

Со временем каждый займет свою нишу в коллективе, но пока никто никого не знает.

Найти общий язык с преподавателями.

Их много и все с разными требованиями. Но в одном точка зрения преподавателей совпадает: успешный студент – самостоятельный и ответственный.

Разобраться в новой ситуации обучения и привыкнуть к ней.

План действий на первое время (до начала учебы):

– узнайте номер группы, в которую Вы зачислены;

– заведите блокнот для важной информации. Перепишите в него расписание занятий;

– принесите с собой ручки разных цветов для удобства ведения конспекта и несколько чистых толстых тетрадей (в зависимости от количества учебных дисциплин в этот день);

– придите немного раньше начала занятий, чтобы не спеша определиться с расположением нужных аудиторий в учебных корпусах. Обычно номера кабинетов трехзначные – первая цифра означает этаж, а последующие – порядковый номер аудитории.

Типичные ошибки студентов-первокурсников:

1. Прогулы.

Кажется, пропустишь одну-две-три пары и ничего не потеряешь. Но это опасное ложное ощущение! В один «прекрасный» момент увидишь, что упустил много и догнать остальных будет очень трудно. Помните – успех складывается из ежедневных усилий!

2. Отчаяние.

Не пасуйте перед трудностями! Как бы трудно не приходилось, не опускайте руки! Не сдал что-то с первого раза – подготовься к пересдаче. А кто сказал, что будет легко?! Тяжело в учении, легко в бою!

И помните, что первый год обучения – самый важный, так как именно в это время происходит формирование основных учебных навыков, закладка базовых знаний. От этого зависит успешность обучения в профессиональном учебном заведении вообще.

Таким образом, на первом курсе нужно как можно больше сил и времени отдавать учебе, чтобы в последующем иметь возможность спокойно, безболезненно сочетать учебу с личной жизнью, досугами и другими сферами жизни. Ваш успех в ваших руках!

Одной из ведущих дисциплин естественнонаучного цикла, которую Вам предстоит изучать на первом и втором курсах, *является математика*. Она создает базу для специальной подготовки, дает возможность творчески решать проблемы современного производства. Кроме того, вооружение навыками самостоятельной творческой работы и самообразования происходит особенно активно в процессе изучения математики. Объясняется это тем, что среди изучаемых Вами на первом курсе дисциплин математика занимает значительную часть времени, причем для овладения ею необходим большой и целеустремленный труд, требуются умственные и волевые усилия, концентрация внимания, активность и систематичность, развитое воображение. Вот почему при обучении математике последовательно и планомерно формируются у студентов рациональные приемы учебной деятельности, умения и навыки умственного труда: планирование своей работы, поиск рациональных путей ее выполнения, критическая оценка результатов.

Основными формами обучения математике в университетах являются *лекции*, *практические занятия*, *самостоятельная работа над учебным материалом*, которая состоит из следующих этапов:

изучения теоретического материала по учебникам, учебным пособиям, конспектам лекций и т.д.;

решения задач и упражнений на практических занятиях;

выполнения домашних заданий, типовых расчетов.

Завершающим этапом изучения отдельных частей курса математики является сдача зачетов и экзаменов в соответствии с учебным планом.

Советы по конспектированию лекций.

Содержание лекции запишите. Запись поможет вам осознать план и логику изложения, осмыслить материал и сосредоточить внимание на основных вопросах. Наличие конспекта лекции позволит вам лучше разобраться в новом материале, додумать и расширить его с помощью учебной литературы.

Записывайте лишь самое главное, не стремитесь зафиксировать всё слово в слово. Практически получается так, что вы, не успев записывать дословно, делаете пропуски, у вас появляются пустые места и, таким образом, упускаете самое главное. Такая запись лишена логического смысла, а потому негодна для использования.

Не стремитесь записывать содержание лекции кратко, настолько кратко, что вы упускаете при этом основные положения.

Не ограничивайтесь записью заголовка, плана и рекомендуемой литературы. Такие записи не отображают основного содержания лекции, и пользоваться ими невозможно.

Мысль преподавателя излагайте своими словами. Такая форма записи позволит вам не только понять услышанное, но и усвоить его. Осмысливая, перерабатывая поступающую информацию, вы можете сохранить устойчивое внимание и предотвратить механическое воздействие.

При конспектировании лекций соблюдайте ряд правил:

1) Учитесь следить за мыслью преподавателя во время изложения нового материала, разделяйте своё внимание между отдельными положениями лекции.

2) Обращайте внимание на тон изложения, интонацию (главные предложения выделяются и произносятся громче).

3) Записи по каждому предмету ведите в отдельной тетради, не пишите на разных листках, которые, как правило, теряются.

Как готовиться к экзаменам.

Как ни странно, но лучший способ хорошо сдать экзамен – это регулярно заниматься в течение года. Тогда материал будет постепенно укладываться в голове, перерабатываться и систематизироваться. Новые знания по изучаемому материалу будут поступать уже на подготовленную почву, а уже имеющиеся в голове под их воздействием будут дополняться и переосмысливаться. И перед экзаменом такой учащийся с удивлением поймёт, что, оказывается, учить – то ничего и не надо, он и так уже всё знает.

Ваш джентльменский набор для сдачи экзаменов должен состоять из *списка билетов*, *конспектов лекций* и *нескольких учебников*. Если «раскопаете» в интернете чьи-то шпаргалки – тоже очень хорошо.

Расписание экзаменов составляется таким образом, чтобы перерыв между двумя экзаменами был не менее 3 дней. Поэтому делите количество свободных дней на количество билетов и начинайте подготовку.

Время подготовки к экзамену надо разумно распределить. Не следует заниматься по много часов без перерывов. Лучше учить блоками – усвоил тему, закрепил её и отдохнул. Затем кратко повторил, что заучил, и – за новую тему. Не стоит заниматься и по ночам, наоборот, готовясь к экзаменам, надо хорошо выспаться, тогда и голова будет работать лучше. Психологи иногда советуют устраивать себе в дни подготовки к экзаменам дробный сон – меньше спать ночью (раньше вставать, а не позже ложиться), но зато спать днём, как в детсадовский «тихий час».

Перед сном можно повторить особо трудный материал. Как известно, хорошо запоминается то, что было выучено последним. Кроме того, во время сна полученные знания будут перерабатываться мозгом, и переходить в долговременную память в спокойной обстановке, не подгоняемые поступающей новой информацией.

Выбирайте в первую очередь самые трудные для себя вопросы, т.к. потом у вас не будет времени их подготовить. То, что знаете хорошо, повторите в самом конце подготовки.

Если любите писать шпаргалки – пишите на здоровье.

Готовить шпаргалки полезно, но пользоваться ими рискованно. Главный смысл подготовки шпаргалок – это систематизация и оптимизация знаний по данному предмету, что само по себе прекрасно – это очень сложная и важная для студента работа, более сложная и важная чем «тупое», «методическое» и «спокойное» поглощение массы (точнее – «кучи») учебной информации. Если студент самостоятельно подготовил такие шпаргалки, то, скорее всего, он и экзамены сдавать будет более уверенно, так как у него уже сформирована общая ориентировка в сложном материале. К сожалению, многие студенты даже в собственных конспектах часто ориентируются очень плохо.

Например, иногда мы проводили экзамены, разрешая пользоваться своими конспектами (и даже учебниками) во время самого ответа. Иногда нескольких секунд было достаточно, чтобы оценить, заглядывал ли студент в свои конспекты (и тем более в книги) при подготовке к данному ответу.

Что делать, если экзамен не сдан?

Первое и главное – не впадать в отчаяние.

Не пытайтесь скандалить с преподавателями, обвиняя его в несправедливой оценке вашего ответа, сохраняйте собственное достоинство.

Следующее рекомендации помогут вам добиться успеха на переэкзаменовке:

– смиритесь с ситуацией, обида – плохой помощник в подготовке;

– относитесь к ситуации как к благоприятной возможности освоить то, в чём вы оказались недостаточно сильны;

– определите, что вы конкретно не знали.

Авторы надеются, что приведенные рекомендации помогут успешно освоить не только математические дисциплины, но и другие предметы во время обучения в высшем учебном заведении.

Теоретический раздел

§1. Linear algebra

1.1 Matrix. The basic definition. Matrix operations.

Definition. A rectangular array of numbers, algebraic symbols or mathematical functions of the species

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

consisting of m rows and n columns is called matrix. Elements of the A matrix are numbered by two indices. Square brackets are also used to designate the matrix. Thus, the element a_{12} belongs to the first line and to the second column. We'll write abbreviated $A = \left(a_{ij}\right),\ i = \overline{1,m},\ j = \overline{1,n}$.

If $m = n$ then the matrix is called the square matrix of order n. For the square matrix, the elements a_{ii} make up the ***main diagonal***. The zero - matrix is called a matrix, all elements of which are zero; it is denoted by the letter *O*. The square matrix, which has all the elements that do not stand on the main diagonal, are zero, is called ***diagonal***. The diagonal matrix, in which all elements of the main diagonal are equal to one, is called an ***identity matrix***. An identity matrix is denoted by the letter *E*. The square matrix is called triangular, if all the elements located on one side of the main diagonal are zero. The matrix derived from this replacement of each of its lines with a column with the same number is called transposed. So, if $A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 5 & \pi \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 4 & \pi \end{pmatrix}$

and $A = A_{2\times3},\ A^T = A^T_{3\times2},\ \left(A^T\right)^T = A$.

The main operations over the matrix are: the addition (subtraction) of the matrix; multiplication of the matrix by real number. Additions (subtractions) are introduced only for matrixes of the same size.

Definition. The sum of the matrices $A+B$, where $A=(a_{ij})$ and $B=(b_{ij})$ are of the same size $m\times n$, is called matrix $C=(c_{ij})$ of the same size $m\times n$, each element is equal to the sum of the corresponding elements of the *A* and *B* matrices:

$$c_{ij}=a_{ij}+b_{ij},\ i=\overline{1,m},\ j=\overline{1,n}. \tag{1.1}$$

Similarly, the concept of the subtraction of the two matrixes is introduced $C=A-B$.

Definition. The product of the matrix *A* on the number $\alpha\in R$ is called the matrix $B_{m\times n}$ such that

$B=\alpha\cdot A=A\cdot\alpha$ and

$$b_{ij}=\alpha a_{ij},\ i=\overline{1,m},\ j=\overline{1,n}.$$

Example 1.1. Find an X matrix that satisfies the condition $X=3A-4E$, then $A=\begin{pmatrix}3 & 2 & -1\\ -4 & 0 & 6\\ 1 & 0 & 0\end{pmatrix}$.

Solution. In this case, $E=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ and, therefore,

$$X=3\cdot\begin{pmatrix}3 & 2 & -1\\ -4 & 0 & 6\\ 1 & 0 & 0\end{pmatrix}-4\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}9 & 6 & -3\\ -12 & 0 & 18\\ 3 & 0 & 0\end{pmatrix}-\begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}=\begin{pmatrix}5 & 6 & -3\\ -12 & -4 & 18\\ 3 & 0 & -4\end{pmatrix}$$

The product of two matrices *A* and *B* could be calculated only if the number of columns of matrix A is equal to the number of lines of the B matrix, i.e. if A is a $m\times n$ size matrix, then B should have a $n\times k$ size.

Definition. The product of the matrix *A* on the matrix *B* is called the matrix *C*, the elements of which are based on the formula.

$$c_{ij}=a_{i1}\cdot b_{1j}+a_{i2}\cdot b_{2j}+\ldots+a_{in}\cdot b_{nj},\quad i=\overline{1,m},\ j=\overline{1,n}, \qquad (1.3)$$

or $c_{ij}=\sum_{s=1}^{n}a_{is}\cdot b_{sj},\quad i=\overline{1,m},\ j=\overline{1,n}$.

If there is a product $A\cdot B$, the product $B\cdot A$ may not exist. It may be that when there is $B\cdot A$. $A\cdot B\neq B\cdot A$.

If $A\cdot B=B\cdot A$, then the *A* and *B* matrix are called permutations (or commuting).

Example 1.2. Find the composition of the matrix $A=\begin{pmatrix}-2 & 4 & 3\\ 1 & -6 & 2\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 0\\ 2 & -3\\ 4 & 5\end{pmatrix}$.

Solution:

$$A\cdot B=\begin{pmatrix}-2 & 4 & 3\\ 1 & -6 & 2\end{pmatrix}\cdot\begin{pmatrix}-1 & 0\\ 2 & -3\\ 4 & 5\end{pmatrix}=\begin{pmatrix}-2\cdot(-1)+4\cdot 2+3\cdot 4 & -2\cdot 0+4\cdot(-3)+3\cdot 5\\ 1\cdot(-1)+(-6)\cdot 2+2\cdot 4 & 1\cdot 0+(-6)\cdot(-3)+2\cdot 5\end{pmatrix}=\begin{pmatrix}22 & 3\\ -5 & 28\end{pmatrix}.$$

Answer: $A\cdot B=\begin{pmatrix}22 & 3\\ -5 & 28\end{pmatrix}$.

Note that the product $B\cdot A$ exists:

$B\cdot A=\begin{pmatrix}-1 & 0\\ 2 & -3\\ 4 & 5\end{pmatrix}\cdot\begin{pmatrix}-2 & 4 & 3\\ 1 & -6 & 2\end{pmatrix}=\begin{pmatrix}2 & -4 & -3\\ -7 & 26 & 0\\ -3 & -14 & 22\end{pmatrix}$, and $A\cdot B\neq B\cdot A$.

1.2 Determinants. Minors and cofactors.

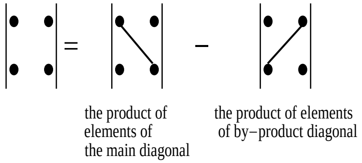
Each square matrix A of order n has the corresponding real number $\det A\left(|A|\ or\ \Delta\right)$, which is called its determinant. Consider the determinants of the 1st, 2nd and 3rd orders.

Let $n=1,\quad A=(a_{11})$. The determinant $\det A = a_{11}$.

Let $n=2,\quad A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Second-order determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}. \tag{1.4}$$

The calculation of the 2nd order determinant is depicted by the scheme:



Let $n=3,\quad A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Third-order determinant

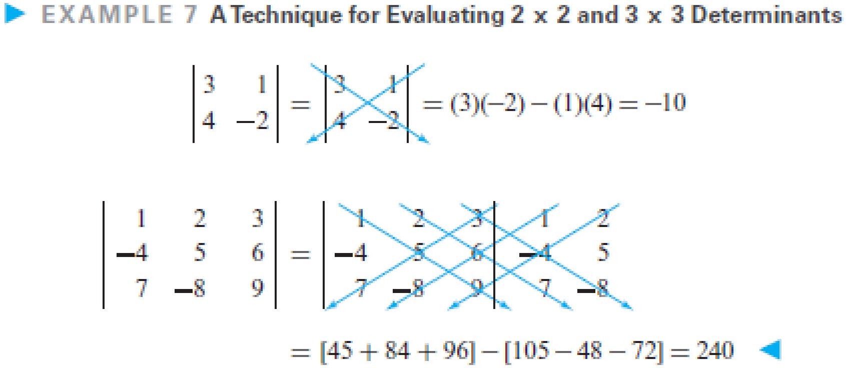
$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{32}a_{23}a_{11}. \tag{1.5}$$

Third-order determinants are usually calculated using the triangle rule (or Sarroys rule). The essence of it is that the determinant in (1.5) consists of three components taken with the sign "+" in the scheme (Figure 1.1, a) and three components taken with the sign "–" on the scheme (Figure 1.1, b).

(+)
a

(–)
b

Fig. 1.1



EXAMPLE 7 A Technique for Evaluating 2 x 2 and 3 x 3 Determinants

$$\begin{vmatrix}3&1\\4&-2\end{vmatrix} = \begin{vmatrix}3&1\\4&-2\end{vmatrix} = (3)(-2)-(1)(4) = -10$$

$$\begin{vmatrix}1&2&3\\-4&5&6\\7&-8&9\end{vmatrix} = \left|\begin{matrix}1&2&3\\-4&5&6\\7&-8&9\end{matrix}\right|\begin{matrix}1&2\\-4&5\\7&-8\end{matrix}$$

$$= [45+84+96]-[105-48-72] = 240$$

Example 1.3. Find the matrix determinant $A = \begin{pmatrix}21 & 3\\ -2 & 4\end{pmatrix}$.

Solution:

According to the formula (1.4) we have:

$$\det A = \begin{vmatrix} 21 & 3 \\ -2 & 4 \end{vmatrix} = 21\cdot 4 - (-2)\cdot 3 = 84 + 6 = 90 .$$

Example 1.4. Find determinant of the matrix $A = \begin{pmatrix} -2 & 1 & -3 \\ 7 & 2 & -1 \\ 5 & 4 & 6 \end{pmatrix}$.

Solution:

According to the formula (1.5) we have:

$$\det A = (-2)\cdot 2\cdot 6 + 7\cdot 4\cdot (-3) + 1\cdot (-1)\cdot 5 - 5\cdot 2\cdot (-3) - 7\cdot 1\cdot 6 - 4\cdot (-1)\cdot (-2) = -133 .$$

Let's give a square matrix of the 4th order. We will select in it arbitrarily s rows and s columns elements standing at the intersection of s rows and s columns form the matrix of order s. Determinant of this matrix is called the ***minor*** of order *s* and is designated by *M*. For example, for the matrix

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ when $s = 2$ choose the second and third lines, the first and fourth columns.

Then the second-order minor will be the determinant

$$M = \begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix}$$

and additional minor will be a determinant.

$$M' = \begin{vmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{vmatrix}.$$

Each elemeht a_{ij} of the 4-th order matrix is a minor of the first order. By removing i-th row and j-th coiumn, we obtain a submatrix of A, having the order (n-1). The determinant of that submatrix is called the minor of the elemeht a_{ij}, which is denoted by M_{ij} .

Definition. The ***cofactor*** of the elemeht a_{ij} is defined as M_{ij} with the sign $(-1)^{i+j}$, it is denoted by the symbol A_{ij}:

$$A_{ij} = (-1)^{i+j} M_{ij} . \tag{1.6}$$

Then let's assume by definition that the determinant of the matrix of the fourth order

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + a_{i4}A_{i4}, \; i = \overline{1,4}. \tag{1.7}$$

The formula (1.7) is called ***the expansion of the determinant fourth order defined by the elements of the i-row***. You can show, that

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + a_{4j}A_{4j}, \; j = \overline{1,4} . \tag{1.8}$$

(***the expansion of the determinant by the elements of the j-column***).

Similarly, one can introduce the concept of a n-order determinant.

Theorem (about the expansion of the determinant by the elements of the rows). The determinant for the n-order's square matrix is equal to the sum of the products of elements of any row of A and the corresponding cofactor, i.e. the formula is fair

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} a_{ik}A_{ik} = \sum_{k=1}^{n} a_{kj}A_{kj} \left(i = \overline{1,n}, \; j = \overline{1,n} \right). \tag{1.9}$$

For

$n=3$ and $i=1$ formula (1.9) will take shape

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}\,, \tag{1.10}$$

where

$$A_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$A_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$A_{13} = (-1)^{1+3} \cdot M_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Here are some of the determinants properties.

The matrix determinant is equal to the transposed matrix determinant, i.e. $\det A = \det A^T$.

The determinant changes the sign to the opposite, if two columns (two rows) of a matrix are interchanged.

The determinant for the matrix O is equal 0.

A determinant with two identical parallel columns or rows is zero.

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$. It is true for any $\lambda \in R$ and any column or row.

If the corresponding elements of the two parallel rows of the determinant are proportional, it is zero.

If you add elements of one row multiplied by a number to the elements of a parallel row of the determinant, the magnitude of the determinant will not change.

Example 1.5. Evaluate the determinant for the matrix A by a cofactor expansion along the first row, $A=\begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}$.

Solution:

Let's use the formula (1.10):

$$\det A=\begin{vmatrix} 1 & -1 & 0 \\ -2 & 3 & 4 \\ 1 & 2 & 2 \end{vmatrix}=1\cdot A_{11}+(-1)\cdot A_{12}+0\cdot A_{13} \text{ ,}$$

where

$$A_{11}=(-1)^{1+1}\cdot\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix}=3\cdot 2-4\cdot 2=-2; \qquad A_{12}=(-1)^{1+2}\cdot\begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix}=-(-2\cdot 2-4\cdot 1)=8;$$

$$A_{13}=(-1)^{1+3}\cdot\begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix}=-2\cdot 2-3\cdot 1=-7. \quad \det A=1\cdot(-2)+(-1)\cdot 8+0\cdot(-7)=-2-8=-10.$$

Note that A_{13} it was possible not to calculate, because $a_{13}=0$.

1.3 Inverse matrix.

Let A be a square matrix of order n

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Definition. The square matrix A is called ***regular*** if its the determiner is not zero, i.e. $\Delta = \det A \neq 0$. If $\Delta = \det A = 0$, then matrix A is called ***singular***.

The matrix A^{-1} is called ***inverse*** to the square matrix A if

$$A^{-1} \cdot A = A \cdot A^{-1} = E, \tag{1.11}$$

where E is a identiy matrix.

Theorem. For the regular matrix A, there is only one an iverse matrix A^{-1}.

The inverse matrix can be found by the next formula:

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}, \tag{1.12}$$

where A_{ij} – cofactor of matrix elements a_{ij} for the matrix A. For a third-order square matrix, the formula (1.12) has the next form:

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}. \tag{1.13}$$

Example 1.6. Find the inverse to the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}$.

Solution:

Since $\Delta = \det A = -10 \neq 0$ (see example 1.5), the matrix A is regular and the inverse matrix exists. We'll find cofactors to the matrix elements for A:

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 1 \cdot (6-8) = -2; \qquad A_{11} = -2;$$

$$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} = -1 \cdot (-4-4) = 8; \qquad A_{12} = 8;$$

$$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} = 1 \cdot (-4-3) = -7; \qquad A_{13} = -7;$$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix} = -1 \cdot (-2-0) = 2; \qquad A_{21} = 2;$$

$$A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 1 \cdot (2-0) = 2; \qquad A_{22} = 2;$$

$$A_{23} = -3; \quad A_{31} = -4; \quad A_{32} = -4; \quad A_{33} = 1.$$

Substituting all the results in (1.13), we get the reverse matrix

$$A^{-1} = -\frac{1}{10}\begin{pmatrix} -2 & 2 & -4 \\ 8 & 2 & -4 \\ -7 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 0{,}2 & -0{,}2 & 0{,}4 \\ -0{,}8 & -0{,}2 & 0{,}4 \\ 0{,}7 & 0{,}3 & 0{,}1 \end{pmatrix}.$$

Example 1.7. Solve the matrix equation $X \cdot \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 50 & -20 \\ 40 & 10 \end{pmatrix}$.

Solution:

Matrix determinant $\Delta = \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = -3 - 2 = -5 \neq 0$. Matrix is regular, we find the cofactors

$$A_{11} = (-1)^{1+1} \cdot |3| = 3; \quad A_{12} = (-1)^{1+2} \cdot |2| = -2; \quad A_{21} = -2; \quad A_{22} = -1.$$

Let's designate $A = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 50 & -20 \\ 40 & 10 \end{pmatrix}$, then the matrix equation will be written in the form of $X \cdot A = B$. Multiply both parts of the last equation on A^{-1} from the right side: $X \cdot A \cdot A^{-1} = B \cdot A^{-1}$. Since $A \cdot A^{-1} = E$, then $X \cdot E = B \cdot A^{-1} \Rightarrow X = B \cdot A^{-1}$. Find the reverse matrix A^{-1} on the formula (1.12) at $n=2$.

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}. \qquad (1.14)$$

Therefore, $A^{-1} = -\frac{1}{5}\begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$.

Finding the matrix

$$X = B \cdot A^{-1}; \quad X = \begin{pmatrix} 50 & -20 \\ 40 & 10 \end{pmatrix} \cdot \left(-\frac{1}{5}\right)\begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}.$$

$$X = -\frac{1}{5}\begin{pmatrix} 50 \cdot 3 + (-20) \cdot (-1) & 50 \cdot (-2) + (-20) \cdot (-1) \\ 40 \cdot 3 + 10 \cdot (-1) & 40 \cdot (-2) + 10 \cdot (-1) \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} 170 & -80 \\ 110 & -90 \end{pmatrix} = \begin{pmatrix} -\frac{170}{5} & \frac{-80}{-5} \\ \frac{110}{-5} & \frac{-90}{-5} \end{pmatrix} = \begin{pmatrix} -34 & 16 \\ -22 & 18 \end{pmatrix}.$$

Answer: $X = \begin{pmatrix} -34 & 16 \\ -22 & 18 \end{pmatrix}$

Note some of the properties of reverse matrix:

$$\det\left(A^{-1}\right) = \frac{1}{\det A};$$

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1};$$

$$\left(A^{-1}\right)^{T} = \left(A^{T}\right)^{-1}.$$

1.4. Matrix rank.

Definition. ***The rank*** of the matrix is called the largest of the orders of its minors, other than zero. If all matrix minors are zero, the rank of the matrix is considered to be zero. Symbol:

rang A; r_A, r .

Definition. ***The basic minor*** of the matrix is called a non-zero minor, the order of which is equal to the rank of the matrix.

For the non-zero matrix there is a basic minor, generally speaking, not the only one.

For the matrix $A = \begin{pmatrix} 2 & -1 & 2 \\ 4 & -2 & 4 \end{pmatrix}$ the rank is equal to1, as there is a minor of the 1st order, different from zero (for example, $|2| = 2 \neq 0$) and all minors of the 2nd order are zero (due to the proportionality of the lines). The basic minor is every minor of the 1st order.

For the matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ the rank is equal to 2, e.i. $r_B = 2$, because $\det B = -2 \neq 0$. The only basic matrix minor coincides with its determinant $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Definition. Minor, edging minor *M* of order k for the matrix A, is called the minor of order (k+1) of this matrix containing minor *M*.

Definition. ***Elementary transformations*** of the matrix will be called the following operations:

multiplying the row (column) of the matrix by a nonzero number;

adding to one row (column) of the other matrix row (column) multiplied by an arbitrary nonzero number;

permutation of two lines (columns) of the matrix.

It is known that in the elementary transformations of the matrix its rank does not change.

Rank of matrix can be found in the following ways.

The method of elementary transformation of finding the rank of the matrix is that any non-zero matrix with the help of elementary transformations can lead to upper triangle form or row echelon form, i.e. to the matrix of the species

$$B = \left(\begin{array}{cccc|ccc} b_{11} & b_{12} & \dots & b_{1r} & b_{1r+1} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2r} & b_{2r+1} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \underline{0} & \underline{0} & \underline{\dots} & \underline{b_{rr}} & b_{rr+1} & \dots & b_{rn} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{array}\right)$$, where is $b_{11}, b_{22}, \dots, b_{rr} -$ nonzero.

Let's cross out zero lines in B. The rank of the received matrix is r - the number of nonzero lines.

Therefore $r_B = r$, and $r_A = r$. The base minor in matrix B is a dedicated minor $\begin{vmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ 0 & b_{22} & \dots & b_{2r} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{rr} \end{vmatrix}$.

The method of fringing minors of the matrix rank calculation is based on the following statement.

Theorem. If matrix A has a minor M of r, different from zero, and all minors bordering minor M (if they exist) are zero, then

$r_A = r$.

To find the rank of matrix A, you need to:

1) Find some minor of the 1st order (i.e. the element of the matrix) different from zero.

If there is no such minor, then the matrix A is zero and $r_A = 0$.

2) Calculate the 2-nd order minors that border minor until there is a minor other than zero. If there is no such minor, then $r_A = 1$; if there is, then $r_A \geq 2$

.Etc..

The process must continue until either all the bordering minors are zero, or the minors of (k+1) order of this matrix do not exist.

In this case, $r_A = k$.

Note that when finding the rank of the matrix in this way is enough at each step to find only one nonzero D minor k-order for $M_{k-1} \neq 0$**.**

Example 1.8. Find the rank of the matrix by the method of elementary transformations: $A = \begin{pmatrix} 1 & -3 & 2 & 5 \\ -2 & 4 & 3 & 1 \\ 0 & -2 & 7 & 11 \\ 7 & -15 & -7 & 2 \\ -1 & 1 & 5 & 6 \end{pmatrix}$.

Solution:

Let's bring the matrix to row echelon form:

$$\begin{pmatrix} 1 & -3 & 2 & 5 \\ -2 & 4 & 3 & 1 \\ 0 & -2 & 7 & 11 \\ 7 & -15 & -7 & 2 \\ -1 & 1 & 5 & 6 \end{pmatrix}^{[1]} \to \begin{pmatrix} 1 & -3 & 2 & 5 \\ 0 & -2 & 7 & 11 \\ 0 & -2 & 7 & 11 \\ 0 & 6 & -21 & -33 \\ 0 & -2 & 7 & 11 \end{pmatrix}^{[2]} \to \left(\begin{array}{cc|cc} 1 & -3 & 2 & 5 \\ \underline{0} & \underline{-2} & 7 & 11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to r_A = 2$$ (this is the number of nonzero lines).

Here are the numbers [1], [2] what's next:

[1] – to the second line add the 1st, multiplied by (–2);

to the 4th line add the 1st, multiplied by (–7);

To the 5th line add the 1st.

[2] –3rd and 5th lines deducted the 2nd line;

to the 4th line added the 2nd, multiplied by 3.

Answer: $r_A = 2$.

Example 1.9. By the method of edging minors to find the rank of the matrix and specify one of the base minors: $A = \begin{pmatrix} \underline{1}| & 2| & 1| & -4 & 5 \\ \underline{3} & \underline{4}| & 0| & -1 & 0 \\ \underline{4} & \underline{6} & \underline{1}| & -5 & 5 \\ 5 & 8 & 2 & -9 & 10 \end{pmatrix}$.

Solution:

Since matrix A has nonzero elements, $r_A \geq 1$. We will choose as a basic $M_1 = |1| = 1$, standing in the top left corner. Let´s move on to the calculation of the 2nd order minor, bordering the chosen M_1. Choose $M_2 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0$, standing in the top left corner.

Since $M_2 \neq 0$, we move on to the calculation of the third order of the minors.

$$M_3^{(1)} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 4 & 6 & 1 \end{vmatrix} = 0,$$

$M_3^{(2)} = \begin{vmatrix} 1 & 2 & -4 \\ 3 & 4 & -1 \\ 4 & 6 & -5 \end{vmatrix} = 0; \quad M_3^{(3)} = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 4 & 0 \\ 4 & 6 & 5 \end{vmatrix} = 0$, since the third line in all these minors is equal to the sum of the first two lines.

Similarly,

$$M_3^{(4)} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 5 & 8 & 2 \end{vmatrix} = 0,$$

$M_3^{(5)} = \begin{vmatrix} 1 & 2 & -4 \\ 3 & 4 & -1 \\ 5 & 8 & -9 \end{vmatrix} = 0; \quad M_3^{(6)} = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 4 & 0 \\ 5 & 8 & 10 \end{vmatrix} = 0$, since the third row is equal to the sum of the second and the first, multiplied by 2.

Since all the minors of the 3rd order, bordering are zero $M_2 \neq 0$, then $r_A = 2$. One of the base minors is $M_2 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

1.5. Linear Equation Systems. Concepts. Solution of non-degenerate linear systems. The matrix method. Kramer Formula.

Definition. The system of m equations with n unknowns variables $x_1, x_2, \ldots, x_n$ is called linear if it has the appearance of

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{cases} \qquad (1.15)$$

where numbers $a_{ij}\left(i = \overline{1,m};\ j = \overline{1,n}\right)$ are called numerical coefficients, and the numbers $b_j\left(i = \overline{1,m}\right)$ – constants.

Definition. A solution of system (1.15) $(c_1, c_2, \ldots, c_n)$ is a set of values of the unknowns $x_1, x_2, \ldots, x_n$ that reduces all equations (1.15) to identities.

If there exist a solution of simultaneous equations then the system is called ***consistent***; otherwise, the system is ***inconsistenit***. Solving the system of equations means figuring out whether it is consisten or not, and if it is consisten, then find all its solutions. Two systems of equations are called equivalent if any solution to one of them is also the solution of the other and vice versa. Equivalent equation systems are produced as a result of the following transformations:

multiplying the system equation by a number other than zero;

adding to one equation another, multiplied by any number; rearranging the two system equations.

The system of equations, in which all free members are zero ($b_1 = b_2 = \ldots = b_m = 0$) is called a ***homogeneous***.

A homogeneous system is always a consisten system, as it has zero solution $x_1 = x_2 = \ldots = x_n = 0$ ***, although it is not necessarily the only one solution.***

The system (1) can be written in a matrix form $AX = B$,

where $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$ – matrix for the system;

$X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$ –***column (or vector-column) of unknown variables***,

$B = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$ – ***cotnsants column***.

The matrix $\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$ is called the extended matrix of the system.

Consider the case of the n equation system with n unknown variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2; \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n, \end{cases} \tag{1.16}$$

or in a matrix form: $AX = B$.

The system (1.16) is called ***consisten*** if the coefficient matrics is regular: $\det A = \Delta \neq 0$.

The consisten system (1.16) n=m has the only one solution that can be found

1) matrix method by formula $X = A^{-1}B$, (1.17)

2) Kramer's formulas: $x_i = \frac{\Delta_i}{\Delta}, \quad i = \overline{1,n}$, (1.18)

Where Δ_i is the matrix determiner derived from the A-pillar matrix by changing its i-th column the column of free members B.

Example 1.10. Solve the equation system $\begin{cases} 2x_1 - x_2 + x_3 = 0, \\ 3x_1 - 2x_2 - x_3 = 5, \\ x_1 + x_2 + x_3 = 6 \end{cases}$ by Kramer's formulas and by matrix method.

Solution:

The system's matrix looks like: $A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

Its determinant $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 7 \neq 0$ as such, the system is consisten and has a single solution.

Using Kramer's formula (1.18), finding $\Delta_1, \Delta_2, \Delta_3$:

$$\Delta_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & -2 & -1 \\ 6 & 1 & 1 \end{vmatrix} = 28; \Delta_2 = \begin{vmatrix} 2 & 0 & 1 \\ 3 & 5 & -1 \\ 1 & 6 & 1 \end{vmatrix} = 35; \Delta_3 = \begin{vmatrix} 2 & -1 & 0 \\ 3 & -2 & 5 \\ 1 & 1 & 6 \end{vmatrix} = -21.$$ Then

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{28}{7} = 4; \quad x_2 = \frac{\Delta_2}{\Delta} = \frac{35}{7} = 5; \quad x_3 = \frac{\Delta_3}{\Delta} = \frac{-21}{7} = -3.$$

Answer: (4; 5; –3).

2) Find

$$A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -1 & 2 & 3 \\ -4 & 1 & 5 \\ 5 & -3 & -1 \end{pmatrix}.$$

Then by formula (1.17)

$$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\frac{1}{7}\begin{pmatrix}-1&2&3\\-4&1&5\\5&-3&-1\end{pmatrix}\begin{pmatrix}0\\5\\6\end{pmatrix}=\frac{1}{7}\begin{pmatrix}28\\35\\-21\end{pmatrix}=\begin{pmatrix}4\\5\\-3\end{pmatrix}.$$

Answer: (4; 5; –3).

1.6. The investigate of Systems of Linear Equations. The Kroneker-Capelli theorem. Method of Gauss.

Consider the system (1.15).

The Kroneker-Capelli theorem. The system of linear equations (1.15) is consistens if and only if the rank of the system matrix is equal to the rank of the extended matrix of the system: $r_A = r_{\tilde{A}}$ **.**

To investigate a system of linear equations means to find out whether it is solvable or not, and for a solvable system to determine whether it has a single solution or an infinite number of solutions.

Three cases are possible:

$r_A < r_{\tilde{A}}$ –the system is inconsisten (i.e. there are no solutions).

$r_A = r_{\tilde{A}} = n$ (n - number of unknown variables) - the system is consistens and has a single solution.

$r_A = r_{\tilde{A}} < n$ –the system is collaborative and has an infinite number of solutions.

For example, the Gauss method (a method of sequential exclusion of unknown variables) can be used to study linear equation systems. With the help of elementary transformations over the rows, the system of *m* linear equations with n unknown variables can be brought to the next form:

$$\left.\begin{aligned} c_{11}x_1+c_{12}x_2+\ldots+c_{1r}x_r+\ldots+c_{1n}x_n&=d_1;\\ c_{22}x_2+\ldots+c_{2r}x_r+\ldots+c_{2n}x_n&=d_2;\\ \ldots\quad\ldots\quad\ldots\quad\ldots&=\ldots\\ c_{rr}x_r+\ldots+c_{rn}x_n&=d_{r;}\\ 0&=d_{r+1};\\ \ldots&=\ldots\\ 0&=d_m,\end{aligned}\right\}\qquad(1.19)$$

where $c_{ii}\neq 0 \quad (i=1, 2, \ldots, r), \quad r\leq n.$

If at least one of the numbers

$d_{r+1}, d_{r+2}, \ldots, d_m$

is different from zero, the system (1.19), and therefore the original system (1.15) are inconsistent

$(r_A = r < r_{\tilde{A}})$.

If $d_{r+1}=d_{r+2}=\ldots=d_{m}=0$ then $r_A=r_{\tilde{A}}$ and the system (1.19) is collaborative. In the case of $r_A=r=r_{\tilde{A}}<n$ unknown $x_1,\ x_2,\ldots,x_r$ are considered to be basic, and $x_{r+1},\ x_{r+2},\ldots,\ x_n$ – free variables.

Basic unknown variables $x_r, x_{r-1},\ldots, x_2, x_1$ leave in the left part of the equations, and the free variables $x_{r+1}, x_{r+2},\ldots, x_n$ transfer to the right part. From equations (1.19) express sequentially basic unknown variables free variables, which are given arbitrary values, receiving a general solution of the system (1.19) ((1.15)) (the system will have an infinite set of solutions).

If $r_A=r=r_{\tilde{A}}=n$ the system (1.19) will have the only one solution that is found, expressing consistently $x_n, x_{n-1},\ldots, x_2, x_1$ through $d_n, d_{n-1},\ldots, d_2, d_1$.

Example 1.11. Gauss method to solve the system

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 = 12; \\ x_1 - 4x_2 + 3x_3 = -22; \\ 3x_1 - x_2 - 2x_3 = 0. \end{cases}$$

Solution:

The system's extended matrix looks like

$$\widetilde{A} = \begin{pmatrix} 2 & 3 & 5 | & 12 \\ 1 & -4 & 3 | & -22 \\ 3 & -1 & -2 | & 0 \end{pmatrix}.$$

By making elementary transformations over the lines of the extended matrix, we get

$$\begin{pmatrix} 2 & 3 & 5 | & 12 \\ 1 & -4 & 3 | & -22 \\ 3 & -1 & -2 | & 0 \end{pmatrix}^{[1]} \rightarrow \begin{pmatrix} 1 & -4 & 3 | & -22 \\ 2 & 3 & 5 | & 12 \\ 3 & -1 & -2 | & 0 \end{pmatrix}^{[2]} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & -4 & 3 | & -22 \\ 0 & 11 & -1 | & 56 \\ 0 & 11 & -11 | & 66 \end{pmatrix}^{[3]} \rightarrow \begin{pmatrix} 1 & -4 & 3 | & -22 \\ 0 & 11 & -1 | & 56 \\ 0 & 0 & -10 | & 10 \end{pmatrix},$$

where the numbers $[1]$, $[2]$, $[3]$ indicate the following operations:

$[1]$ –first and second lines swapped; $[2]$ – to the second line added the first, multiplied by (–2); to the third added the first, multiplied by (–3); $[3]$ –to the third line added the second, multiplied by (–1).

This matrix corresponds to the system

$$\begin{cases} x_1 - 4x_2 + 3x_3 = -22; \\ 11x_2 - x_3 = 56; \\ -10x_3 = 10. \end{cases}$$

From here we find

$$x_3 = \frac{10}{-10} = -1; \quad 11x_2 = 56 + x_3; \quad x_2 = \frac{56-1}{11} = 5;$$

$$x_1 = -22 + 4x_2 - 3x_3; \quad x_1 = 1.$$

Answer: $x_1 = 1, \quad x_2 = 5, \quad x_3 = -1.$

Example 1.12. Solve the system

$$\begin{cases} x_1 + x_2 - x_3 = -4; \\ x_1 + 2x_2 - 3x_3 = 0; \\ -2x_1 - 2x_3 = 16. \end{cases}$$ by Gauss method.

Solution:

The system's extended matrix looks like

$$\tilde{A} = \begin{pmatrix} 1 & 1 & -1| & -4 \\ 1 & 2 & -3| & 0 \\ -2 & 0 & -2| & 16 \end{pmatrix}$$

With the help of elementary transformations, we will bring the matrix to the row echelon form:

$$\begin{pmatrix} 1 & 1 & -1| & -4 \\ 1 & 2 & -3| & 0 \\ -2 & 0 & -2| & 16 \end{pmatrix}^{[1]} \to \begin{pmatrix} 1 & 1 & -1| & -4 \\ 0 & 1 & -2| & 4 \\ 0 & 2 & -4| & 8 \end{pmatrix}^{[2]} \to \begin{pmatrix} |\overline{1} & \overline{1}| & -1| & -4 \\ |\underline{0} & \underline{1}| & -2| & 4 \\ 0 & 0 & 0| & 0 \end{pmatrix},$$

where $[1], [2]$ the numbers indicate the following operations:

$[1]$ – from the second row we subtract the first row, to the 3rd line add the 1st, multiplied by 2;

$[2]$ – to the 3rd line add the 2nd, multiplied by (– 2).

Let's get that $r_A = r_{\tilde{A}} = 2 < n = 3$. Consequently, the system is collaborative and has an infinite number of solutions. x_1, x_2 – basic unknown variables, x_3 – free variable. The number of basic unknown variables equals $r_A = 2$; number of free unknown variables equals $n - r_A = 1$. Let's write down a system of equations corresponding to the extended matrix:

$$\begin{cases} x_1 + x_2 - x_3 = -4; \\ \qquad x_2 - 2x_3 = 4. \end{cases}$$

On the left side of the equations will leave only the basic unknowns:

$$\begin{cases} x_1 + x_2 = -4 + x_3; \\ \qquad x_2 = 4 + 2x_3. \end{cases}.$$

Substituting x_2 the expression for the 1st equation, we get $x_1 = -x_3 - 8$.

Suppose that $x_3 = C, C \in R$ then the solution system will have the next form:

$$x_1 = -C - 8; \quad x_2 = 4 - 2C; \quad x_3 = C\ .$$

Answer: $(-C - 8; 4 - 2C; C), \quad C \in R$.

Example 1.13. Solve the system by using the Gauss method.

$$\begin{cases} x_1 + x_2 - x_3 = -4; \\ x_1 + 2x_2 - 3x_3 = 0; \\ -2x_1 - 2x_3 = 3. \end{cases}$$

Solution:

Transform to the row echelon form the extended matrix of the system:

$$\tilde{A} = \begin{pmatrix} 1 & 1 & -1| & -4 \\ 1 & 2 & -3| & 0 \\ -2 & 0 & -2| & 3 \end{pmatrix}^{[1]} \rightarrow \begin{pmatrix} 1 & 1 & -1| & -4 \\ 0 & 1 & -2| & 4 \\ 0 & 2 & -4| & -5 \end{pmatrix}^{[2]} \rightarrow \begin{pmatrix} 1 & 1 & -1| & -4 \\ 0 & 1 & -2| & 4 \\ 0 & 0 & 0| & -13 \end{pmatrix},$$

[1] – we're subtract 1-st row from the row 2;

– to the 3rd row add the 1-st row, multiplied by the number 2;

[2] – to the third row add the 2-nd row, multiplied by (–2).

Because $r_A = 2 \neq r_{\tilde{A}} = 3$ the system is inconsisten.

Answer: the system is inconsisten.

§2. Vectors

2.1 Vectors in the plane.

Some of the things we measure are determined by their magnitudes. For example, mass, length, time are described by number and name an appropriate unit of measure. On the other hand, we need more information to describe a force, a velocity or displacement. We need to know also direction. To describe a body´ s displacement, we have to say in what direction it moves as well as how far.

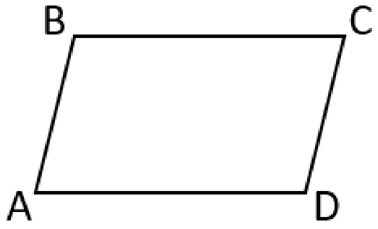
Quantities that have direction as well as magnitude are usually represented by arrows that point in the direction of the action and whose lengths give the magnitude of the action.

Definition. Vector is the directed line segment.

Further, the vector $\bar{a}$ is equal to the vector $\bar{b}$, if they have the same length, parallel, and point in the same direction.

Vectors are usually described with single roman letters with arrows over them: $\vec{a}$ or $\bar{a}$, $\vec{b}$ or $\bar{b}$ and so on. The vector defined by the directed line segment from point A to point B is written as $\overline{AB}$. If the vector $\bar{a}$ is parallel to the vector $\bar{b}$, then we say that $\bar{a}$ is collinear to $\bar{b}$.

Example. Consider the parallelogram:



$\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$.

Moreover, $\overline{AB} = \overline{DC}$ and $\overline{BC} = \overline{AD}$.

Scalar multiples.

We multiply a vector by a positive real number by multiplying its length by the number. To multiply a vector by 3, we triple its length.

We multiply a vector by a negative number by reversing the vector´s direction and multiplying the length by the number´s absolute value. If *c* is a non-zero real number and $\bar{v}$ is a vector, the direction of $c\bar{v}$ agrees with that of $\bar{v}$ if *c* is positive and is opposite to that of $\bar{v}$ if *c* is negative. We call real number scalars and call multiples like *cv* scalar multiples of $\bar{v}$ (fig. 2.1).

Further, multiplying a vector by zero produces the zero vector $\bar{0}$, consisting of points that are degenerate line segments of zero length. Unlike other vectors, the vector $\bar{0}$ has no direction.

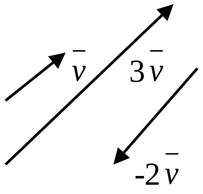


Fig. 2.1. Scalar multiples of $\bar{v}$.

Geometric addition.

Two nonzero vectors $\overline{v_1}$ and $\overline{v_2}$ can be added geometrically by drawing a representative of $\overline{v_1}$, say from A to B as in fig. 2.2, and then a representative of $\overline{v_2}$ starting from the terminal point B of $\overline{v_1}$. In fig. 2.2 , $\overline{v_2} = \overline{BC}$. The sum $\overline{v_1} + \overline{v_2}$ is the vector represented by the arrow from the initial point A of $\overline{v_1}$ to the terminal point C of $\overline{v_2}$.

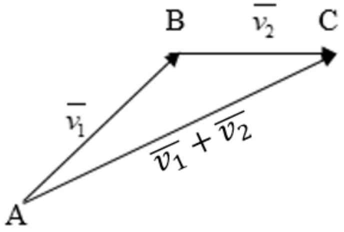


Fig. 2.2. The vector $\overline{AB} + \overline{BC} = \overline{AC}$.

That is, if $\overline{v_1} = \overline{AB}$ and $\overline{v_2} = \overline{BC}$, then $\overline{v_1} + \overline{v_2} = \overline{AB} + \overline{BC} = \overline{AC}$.

The vector $\overline{v_1} + \overline{v_2}$ is given by the diagonal of the parallelogram determined by $\overline{v_1}$ and $\overline{v_2}$. Therefore, the rule is called parallelogram law of addition.

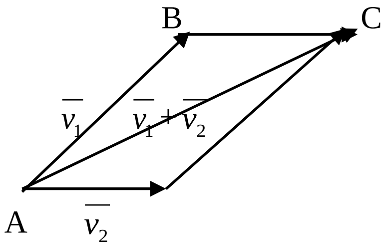


Fig. 2.3. Parallelogram law.

Components.

When a vector $\bar{v}$ can be written as sum $\bar{v} = \overline{v_1} + \overline{v_2}$ of two nonparallel vectors, the vectors $\overline{v_1}$ and $\overline{v_2}$ are said to be components of $\bar{v}$.

The algebra of vector is based on representing each vector in terms of components parallel to the Cartesian coordinate axes and writing each component as an appropriate multiple of a basic vector of length 1. The basic vector in the positive x – direction is the vector $\bar{i}$ determined by the directed line segment that starts from (0; 0) to (1;0). The basic vector in the positive y – direction is the vector $\bar{j}$ determined by the directed line segment from (0;0) to (0;1). Then $a\bar{i}(a \in R)$, represents a vector of length $|a|$ parallel to the x-axis, pointing to the right if $a > 0$ and to the left if $a < 0$. Similarly, $b\bar{j}$ is a vector of length $|b|$ parallel to the y-axis, pointing up if $b > 0$ and down if $b < 0$

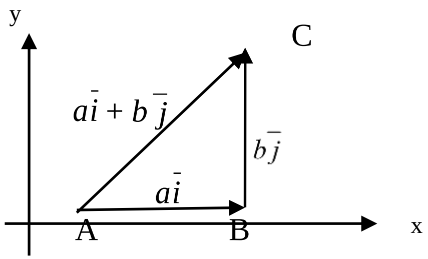


Fig. 2.4. Figure shows a vector $\bar{v} = \overline{AC}$ resolved into its $\bar{i}$ - and $\bar{j}$ - components as the sum $\bar{v} = a\bar{i} + b\bar{j}$.

Definitions.

If $\bar{v} = a\bar{i} + b\bar{j}$, the vectors $a\bar{i}$ and $b\bar{j}$ are the vector components of $\bar{v}$ in the direction of $\bar{i}$ and $\bar{j}$. The numbers *a* and *b* are the scalar components of $\bar{v}$ in the directions of $\bar{i}$ and $\bar{j}$.

Definition. Equality of vectors (Algebraic definition).

$a_1\bar{i} + b_1\bar{j} = a_2\bar{i} + b_2\bar{j} \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

Two vectors are equal if and only if their scalar components in the directions of $\bar{i}$ and $\bar{j}$ are identical.

Algebraic addition.

If $\overline{v_1} = a_1\bar{i} + b_1\bar{j}$ and $\overline{v_2} = a_2\bar{i} + b_2\bar{j}$, then $\overline{v_1} + \overline{v_2} = (a_1 + a_2)\bar{i} + (b_1 + b_2)\bar{j}$.

Example. $(3\bar{i} - 2\bar{j}) + (5\bar{i} - \bar{j}) = \bar{i} \cdot (3+5) + \bar{j} \cdot (-2-1) = 8\bar{i} - 3\bar{j}$.

Subtraction.

The negative of vector $\bar{v}$ is the vector $-\bar{v} = (-1) \cdot \bar{v}$. It has the same length as $\bar{v}$ but points in the opposite direction. To subtract a vector $\overline{v_2}$ from a vector $\overline{v_1}$, we add $-\overline{v_2}$ to $\overline{v_1}$. It is shown geometrically on the Figure 2.5.

If $\overline{v_1} = a_1\bar{i} + b_1\bar{j}$ and $\overline{v_2} = a_2\bar{i} + b_2\bar{j}$, then $\overline{v_1} - \overline{v_2} = (a_1 - a_2)\bar{i} + (b_1 - b_2)\bar{j}$.

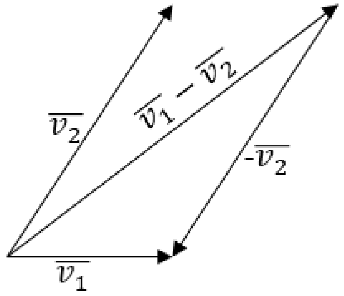


Fig. 2.5.

Example. $(3\bar{i}+2\bar{j})-(5\bar{i}-7\bar{j})=\bar{i}\cdot(3-5)+\bar{j}\cdot(2+7)=-2\bar{i}+9\bar{j}$.

Furthermore, the vector $\overline{P_1P_2}=\overline{OP_2}-\overline{OP_1}$. $P_1(x_1;y_1)$, $P_2(x_2;y_2)$. $\overline{OP_1}=x_1\bar{i}+y_1\bar{j}$, $\overline{OP_2}=x_2\bar{i}+y_2\bar{j}$. Therefore, $\overline{P_1P_2}=\bar{i}(x_2-x_1)+\bar{j}(y_2-y_1)$.

Example. $P_1(1;5)$, $P_2(3;-7)$.

$\overline{P_1P_2}=(3-1;-7-5)=(2;-12)=2\bar{i}-12\bar{j}$.

Magnitude.

The magnitude or length of $\bar{v}=a\bar{i}+b\bar{j}$ is $|\bar{v}|=\sqrt{a^2+b^2}$.

It is follows from the Pythagorean theorem, as on Figure 2.6.

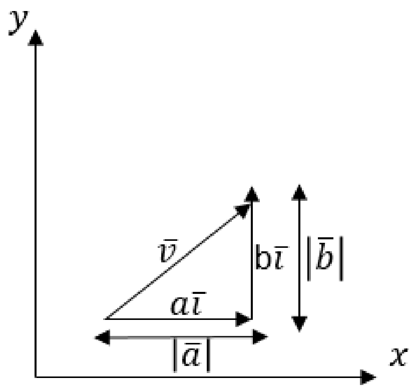
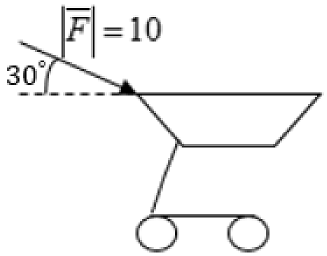


Fig. 2.6. The length of $\bar{v}$ is $\sqrt{a^2+b^2}$.

Example. You push a loaded supermarket cart by applying 10 N force $\bar{F}$ that makes a 30° angle with the horizontal. Resolve $\bar{F}$ into its horizontal and vertical components. (The horizontal component is the effective force in the direction of motion. The vertical component just adds weight to the cart.



Solution.

We draw a vector triangle for $\overline{F} = a\bar{i} + b\bar{j}$ (Figure 2.7).

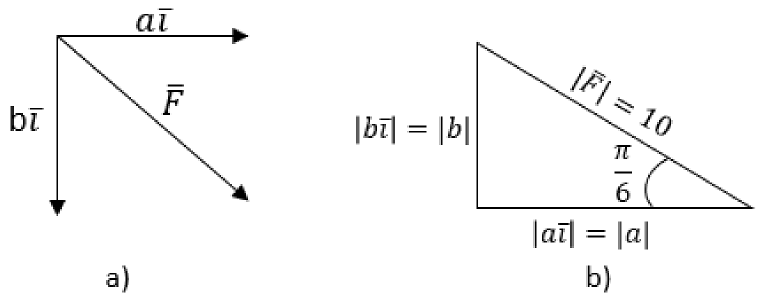


Fig.2.7.

a) $\overline{F} = a\bar{i} + b\bar{j}$

b) $|\bar{b}| = |\overline{F}| \cdot \sin 30^\circ = 5$

$|\bar{a}| = |\bar{F}| \cdot \cos 30^\circ = 10 \cdot \frac{\sqrt{3}}{2} = 5\sqrt{3}$.

The component $b < 0$, because $Oy \uparrow\downarrow b\bar{j}$. So, $\bar{F} = 5\sqrt{3}\bar{i} - 5\bar{j}$.

Scalar multiplication.

Scalar multiplication can be accomplished component by component. If c is a scalar and $\bar{v} = a\bar{i} + b\bar{j}$ is a vector, then $c\bar{v} = c \cdot \left(a\bar{i} + b\bar{j}\right) = (ca)\bar{i} + (cb)\bar{j}$.

The length of $c\bar{v}$ is $|c|$ times the length of v:

$$|c\bar{v}| = |ca\bar{i} + cb\bar{j}| = \sqrt{(ca)^2 + (cb)^2} = \sqrt{c^2(a^2 + b^2)} = \sqrt{c^2} \cdot \sqrt{a^2 + b^2} = |c| \cdot |\bar{v}|.$$

If $c \in R$ and $\bar{v}$ is a vector, then $|c\bar{v}| = |c| \cdot |\bar{v}|$.

Example. $\bar{v} = 3\bar{i} + 5\bar{j}$, $c = 3$.

$$c\bar{v} = 3 \cdot 3\bar{i} + 3 \cdot 5\bar{j} = 9\bar{i} + 15\bar{j}.$$

The zero vector.

The zero vector is the vector $\bar{0} = 0\bar{i} + 0\bar{j}$. It is the only vector whose length is zero, as we can see from the fact that $|a\bar{i} + b\bar{j}| = \sqrt{a^2 + b^2} = 0 \Leftrightarrow a = 0$ and b = 0.

Unit vectors.

Any vector whose length is 1 is a unit vector. The vectors $\bar{i}$ and $\bar{j}$ are unit vectors.

$$\left|\bar{i}\right| = \left|1\cdot\bar{i}+0\cdot\bar{j}\right| = \sqrt{1^2+0^2} = 1.\ \left|\bar{j}\right| = \left|0\cdot\bar{i}+1\cdot\bar{j}\right| = \sqrt{0^2+1^2} = 1.$$

If $\bar{u}$ is the unit vector obtained by rotating $\bar{i}$ through an angle φ in the positive direction, then $\bar{u}$ has a horizontal component $\cos\varphi$ and vertical component $\sin\varphi$ (fig. 2.8).

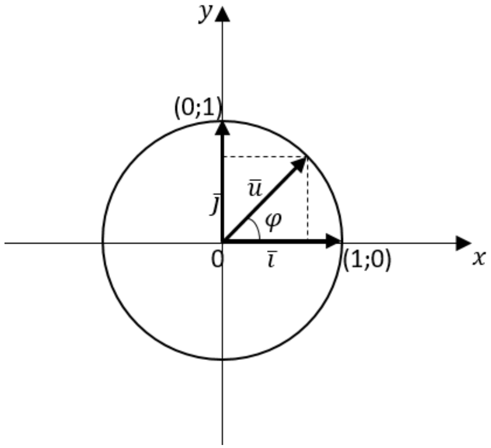


Fig. 2.8. The unit vector that makes an angle of measure φ with the positive

x- axis. Every unit vector in the plane has the form $\bar{u} = \cos\varphi\cdot\bar{i}+\sin\varphi\cdot\bar{j}$ for some φ.

Length and direction.

If $\bar{v} \neq \bar{0}$, then $\left|\frac{\bar{v}}{|\bar{v}|}\right| = \frac{1}{|\bar{v}|} \cdot |\bar{v}| = 1$, so $\frac{\bar{v}}{|\bar{v}|}$ is a unit vector in the direction of $\bar{v}$.

We can therefore express $\bar{v}$ in terms of its two important features, length and direction, by writing $\bar{v} = |\bar{v}| \cdot \left(\frac{\bar{v}}{|\bar{v}|}\right)$. If $\bar{v} \neq \bar{0}$, then:

1. $\frac{\bar{v}}{|\bar{v}|}$ is a unit vector in the direction of $\bar{v}$.

2. The equation $\bar{v} = |\bar{v}| \cdot \frac{\bar{v}}{|\bar{v}|}$ expresses $\bar{v}$ in terms of its length and direction.

Example. Express $-3\bar{i} + 4\bar{j}$ as a product of its length and direction.

Solution. Length of $\bar{v}$: $|\bar{v}| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$.

Direction of $\bar{v}$: $\frac{\bar{v}}{|\bar{v}|} = -\frac{3}{5}\bar{i} + \frac{4}{5}\bar{j}$. So, $\bar{v} = -3\bar{i} + 4\bar{j} = 5 \cdot \left(-\frac{3}{5}\bar{i} + \frac{4}{5}\bar{j}\right)$.

3.1 Cartesian coordinates.

To locate points in space, we use three mutually perpendicular coordinate axes, arranged as in Figure 3.1.

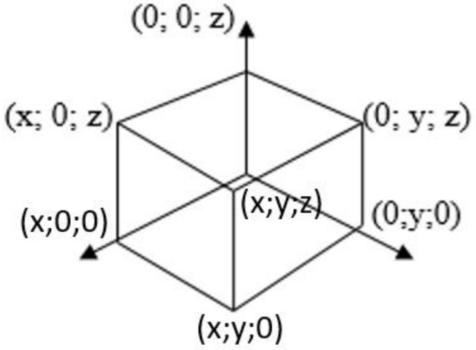


Fig. 3.1. The Cartesian coordinates system.

To locate points in space, we use three mutually perpendicular coordinate axes, arranged as in Figure 3.1.

The Cartesian coordinates (x; y; z) of a point P in space are numbers at which the planes through P perpendicular to the axes cut the axes.

Points on the x-axes have y – and z - coordinate equal to zero. That is, they have coordinate of the form (x; 0; 0). Similarly, points on the y-axes have coordinate of the form (0; y; 0). Points on the z-axes have coordinate of the form (0; 0; z).

The planes determined by the coordinate axes are the xy-plane, whose standard equation is $z = 0$; the yz-plane, whose standard equation is $x = 0$; and the xz-plane, whose standard equation is $y = 0$.The three coordinate planes $x = 0,\ y = 0,\ z = 0$ divide space into eight cells called octants. The octant in which the point coordinates are all positive is called the first octant.

Cartesian coordinates for space are also called rectangular coordinates because the axes that define them meet at right angles.

In the following examples, we match coordinate equation and inequalities with the sets of points they define in space.

Example.

Defining equations and inequalities	Verbal description
$z \geq 0$	The half-space consisting of the points z and above the xy-plane
$x = -2$	The plane perpendicular to the x-axes at $x = -2$. This plane lies parallel to the yz-plane and 2 units behind it.
$x \geq 0,\ y \geq 0,\ z \geq 0$	The first octant.
$-2 \leq y \leq 2$	The layer between the planes $y = -2$ and $y = 2$.
$y = 3,\ z = -2$	The line in which the planes $y = 3$ and $z = -2$ intersect. Alternatively, the line though the point (0; 3;-2) parallel to the x-axis.

Example. What points P(x; y; z) satisfy the equations $x^2 + y^2 = 9$ and $z = 4$.

Solution. The points lie in the horizontal plane $z = 4$ and, in the plane, make up the circle $x^2 + y^2 = 9$. We call this set of points the circle $x^2 + y^2 = 9$ in the plane $z = 4$

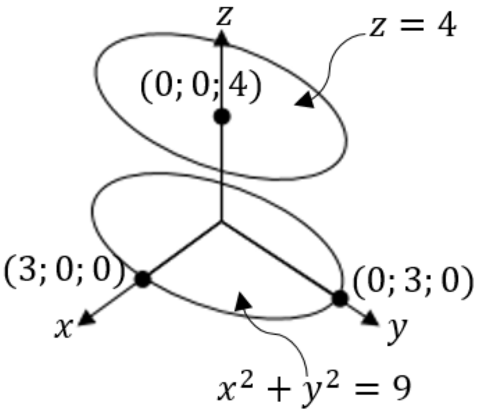


Fig. 3.2. The circle on the plane $z = 4$.

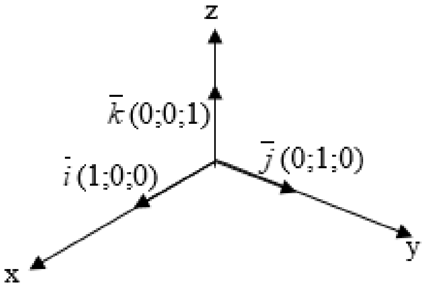
Vector in Space.

The sets of equivalent directed line segments that we use to represent forces, displacements, and velocities in space are called vectors, just as in the plane. The same rules of addition, subtraction, and scalar multiplication apply.

The vector represented by the directed line segments from the origin to the points (1;0;0), (0;1;0) and (0;0;1) are the basic vectors.

The position vector $\bar{r}$ from the origin O to the typical point P(x;y;z) is

$\bar{r} = \overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$.



Definition. For any vectors $\overline{v_1} = x_1\overline{i} + y_1\overline{j} + z_1\overline{k}$ and $\overline{v_2} = x_2\overline{i} + y_2\overline{j} + z_2\overline{k}$, $\overline{v_1} + \overline{v_2} = (x_1 + x_2)\overline{i} + (y_1 + y_2)\overline{j} + (z_1 + z_2)\overline{k}$

$\overline{v_1} - \overline{v_2} = (x_1 - x_2)\overline{i} + (y_1 - y_2)\overline{j} + (z_1 - z_2)\overline{k}$.

The vector between two points.

We express the vector $\overline{P_1P_2}$ in terms of the coordinates of P_1 and P_2 in the next form:

If $P_1(x_1; y_1)$, $P_2(x_2; y_2)$ then $\overline{P_1P_2} = (x_2 - x_1;\ y_2 - y_1)$.

The vector from $P_1(x_1; y_1; z_1)$ to $P_2(x_2; y_2; z_2)$ is $\overline{P_1P_2} = (x_2 - x_1)\overline{i} + (y_2 - y_1)\overline{j} + (z_2 - z_1)\overline{k}$.

The length(magnitude) of $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ is $|\overline{a}| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$.

Scalar multiplication.

If $c \in R$ and $\overline{a} = x_1\overline{i} + y_1\overline{j} + z_1\overline{k}$ then $c\overline{a} = (cx_1)\overline{i} + (cy_1)\overline{j} + (cz_1)\overline{k}$.

Example. If $c = 3$, $\overline{a} = 3\overline{i} - 2\overline{j} + 5\overline{k}$ then $c\overline{a} = 3\overline{a} = 9\overline{i} - 6\overline{j} + 15\overline{k}$.

The zero vector.

The zero vector $\overline{0} = 0\overline{i} + 0\overline{j} + 0\overline{k}$. $|\overline{0}| = 0$.

The vector $\overline{0}$ has no direction.

Unit vector.

A unit vector in space is a vector of length 1.

$|\overline{i}| = \sqrt{1^2 + 0^2 + 0^2} = 1$, $|\overline{j}| = \sqrt{0^2 + 1^2 + 0^2} = 1$, $|\overline{k}| = \sqrt{0^2 + 0^2 + 1^2} = 1$. Therefore, $\overline{i}$, $\overline{j}$ and $\overline{k}$ are unit vectors.

Magnitude and direction.

If $\overline{a} \neq 0$ then $\dfrac{\overline{a}}{|\overline{a}|}$ is a unit vector in the direction of $\overline{a}$. Therefore, $\overline{a} = |\overline{a}| \cdot \dfrac{\overline{a}}{|\overline{a}|}$.

This equation express $\bar{a}$ as a product of its magnitude and direction.

Example. Find a unit vector $\bar{u}$ in the direction from $P_1(1;0;2)$ to $P_2(2;2;4)$.

Solution. $\overline{P_1P_2} = (2-1;\ 2-0;4-2) = (1;2;2)$. $\left|\overline{P_1P_2}\right| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$.

$$\frac{\overline{P_1P_2}}{\left|\overline{P_1P_2}\right|} = \left(\frac{1}{3};\frac{2}{3};\frac{2}{3}\right) = \frac{1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k} = \bar{u}.$$

Distance in space.

The distance in space between two points P_1 and P_2 is the length of $\overline{P_1P_2}$.

$$\left|\overline{P_1P_2}\right| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

Example. The distance between $P_1(3;0;-2)$ and $P_2(5;-3;4)$ is $\left|\overline{P_1P_2}\right| = \sqrt{(5-3)^2 + (-3-0)^2 + (4-(-2))^2} = \sqrt{2^2+3^2+6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$.

Midpoints.

The coordinates of the midpoint of a line segment are found by averaging.

The midpoint M of the line segment joining points $P_1(x_1;y_1;z_1)$ and $P_2(x_2;y_2;z_2)$ is the point

$M\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2};\frac{z_1+z_2}{2}\right)$. To see why, observe that

$$\overline{OM} = \overline{OP_1} + \frac{1}{2}\overline{P_1P_2} = \overline{OP_1} + \frac{1}{2}(\overline{OP_2} - \overline{OP_1}) = \frac{1}{2}\overline{OP_1} + \frac{1}{2}\overline{OP_2} =$$

$$= \frac{x_1+x_2}{2}\bar{i} + \frac{y_1+y_2}{2}\bar{j} + \frac{z_1+z_2}{2}\bar{k}.$$

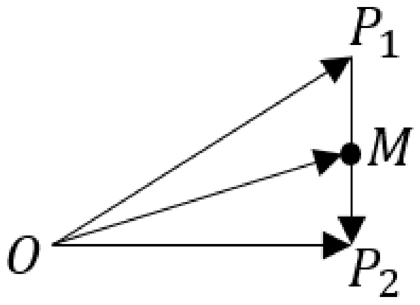


Fig. 3.3. The point M is the midpoint for the line segment P_1P_2.

3.2 Dot products.

We now introduce scalar product, the first of two methods we will learn for multiplying vectors together. The product is called scalar product because the multiplication results in a scalar, not a vector.

When two nonzero vectors $\overline{a}$ and $\overline{b}$ are placed so their initial points coincide, they form an angle φ of measure $0 \le \varphi \le \pi$ (fig. 3.4).

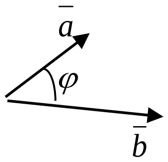


Fig. 3.4. This angle is called the angle between $\overline{a}$ and $\overline{b}$.

Definition. The scalar product (dot product) $\overline{a} \cdot \overline{b}$ of vectors $\overline{a}$ and $\overline{b}$ is a number

$\overline{a} \cdot \overline{b} = |\overline{a}| \cdot |\overline{b}| \cdot \cos\varphi$, where φ is the angle between $\overline{a}$ and $\overline{b}$.

Since the sign of $\bar{a}\cdot\bar{b}$ is determined by $\cos\varphi$, the scalar product is positive if the angle between the vectors is acute, negative if the angle is obtuse.

$$\bar{a}\cdot\bar{a}=|\bar{a}|\cdot|\bar{a}|\cdot\cos 0=|\bar{a}|\cdot|\bar{a}|\cdot 1=|\bar{a}|^2 .$$

$$\bar{i}\cdot\bar{i}=1,\ \bar{j}\cdot\bar{j}=1,\ \bar{k}\cdot\bar{k}=1,\ \bar{i}\cdot\bar{j}=0,\ \bar{i}\cdot\bar{k}=0,\ \bar{j}\cdot\bar{k}=0 .$$

Therefore, if $\bar{a}=x_1\bar{i}+y_1\bar{j}+z_1\bar{k}$ and $\bar{b}=x_2\bar{i}+y_2\bar{j}+z_2\bar{k}$ then

$$\bar{a}\cdot\bar{b}=x_1x_2\bar{i}\cdot\bar{i}+x_1y_2\bar{i}\cdot\bar{j}+x_1z_2\bar{i}\cdot\bar{k}+y_1x_2\bar{j}\cdot\bar{i}+y_1y_2\bar{j}\cdot\bar{j}+y_1z_2\bar{j}\cdot\bar{k}+z_1x_2\bar{k}\cdot\bar{i}+$$

$$+z_1y_2\bar{k}\cdot\bar{j}+z_1z_2\bar{k}\cdot\bar{k}=x_1x_2+y_1y_2+z_1z_2 .$$

The angle between two nonzero vectors $\bar{a}$ and $\bar{b}$ is $\varphi=\arccos\left(\dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|}\right)$.

Example. Find the angle between $\bar{a}=2\bar{i}-3\bar{j}+5\bar{k}$ and $\bar{b}=\bar{i}-\bar{j}+\bar{k}$.

$$\phi=\arccos\left(\frac{2\cdot 1+(-3)\cdot(-1)+5\cdot 1}{\sqrt{2^2+(-3)^2+5^2}\cdot\sqrt{1^2+(-1)^2+1^2}}\right)=\arccos\left(\frac{2+3+5}{\sqrt{4+9+25}\cdot\sqrt{1+1+1}}\right)=$$

$$=\arccos\frac{10}{10{,}68}=\arccos 0{,}94=20{,}56^\circ .$$

$$=\arccos\frac{10}{\sqrt{38}\cdot\sqrt{3}}=$$

Laws of the scalar product.

From the equation $\bar{a}\cdot\bar{b}=x_1x_2+y_1y_2+z_1z_2$, we can see that $\bar{a}\cdot\bar{b}=\bar{b}\cdot\bar{a}$ (commutative law).

Also, if $c\in R$ then $(c\cdot\bar{a})\cdot\bar{b}=\bar{a}\cdot(c\cdot\bar{b})=c\cdot(\bar{a}\cdot\bar{b})$.

Further, $\bar{a}\cdot(\bar{b}+\bar{c})=\bar{a}\cdot\bar{b}+\bar{a}\cdot\bar{c}$ (distributive law).

If we combine this law with (1), we obtain that $(\bar{a}+\bar{b})\cdot\bar{c}=\bar{a}\cdot\bar{c}+\bar{b}\cdot\bar{c}$.

Orthogonal vectors.

Two nonzero vectors $\bar{a}$ and $\bar{b}$ are orthogonal if the angle between them is $\frac{\pi}{2}$. For such vectors, we automatically have $\bar{a}\cdot\bar{b}=0$ because $\cos\left(\frac{\pi}{2}\right)=0$. The converse is also true. If $\bar{a}$ and $\bar{b}$ are nonzero vectors with $\bar{a}\cdot\bar{b}=0$ then $\cos\varphi=0$ and $\varphi=\frac{\pi}{2}$.

Nonzero vectors $\bar{a}$ and $\bar{b}$ are orthogonal if and only if $\bar{a}\cdot\bar{b}=0$.

Example. $\bar{a}=2\bar{i}-\bar{j}+3\bar{k}$ and $\bar{b}=5\bar{i}+28\bar{j}+6\bar{k}$ are orthogonal because $\bar{a}\cdot\bar{b}=10-28+18=0$.

The scalar projection of $\bar{b}=\overline{PQ}$ into nonzero vector $\bar{a}=\overline{PS}$ is the scalar $\operatorname{proj}_{\bar{a}}\bar{b}$ determined by dropping a perpendicular from Q to the line PS(fig. 3.5).

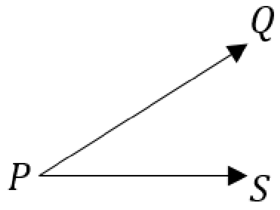


Fig. 3.5. $\operatorname{proj}_{\bar{a}}\bar{b}$.

If the angle φ between $\bar{a}$ and $\bar{b}$ is acute, $\operatorname{proj}_{\bar{a}}\bar{b}>0$. If the angle φ between $\bar{a}$ and $\bar{b}$ is obtuse, then $\operatorname{proj}_{\bar{a}}\bar{b}<0$.

In any case, $\operatorname{proj}_{\bar{a}}\bar{b}=|\bar{b}|\cdot\cos\varphi=b\cdot\frac{\bar{a}}{|\bar{a}|}$.

Work. As you know from high school, the work A done by a constant force of magnitude F in moving an object through a distance d is calculated as $A=F\cdot d\cdot\cos\varphi$. Here φ is the angle between the direction of force $\bar{F}$ and the displacement $\bar{S}$. Therefore, $A=\bar{F}\cdot\bar{S}$.

Cross products.

In space, we need to be able to describe how a plane is tilting. We accomplish this by multiplying two vectors in the plane together to get a third vector perpendicular to the plane. The direction of this third vector tells us the «inclination» of the plane. The product we use to multiply the vectors together is the vector or cross product. Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows and orbital mechanics. This section presents the mathematical properties for cross product.

We start with two nonzero vectors $\bar{a}$ and $\bar{b}$ in space. If $\bar{a}$ and $\bar{b}$ are not parallel, they determine a plane. We select a unit vector $\bar{n}$ perpendicular to the plane by the right-hand rule. This means we choose $\bar{n}$ to be the unit(normal) vector that points the way your right thumb points when your finger curl through the angle φ from $\bar{a}$ and $\bar{b}$.

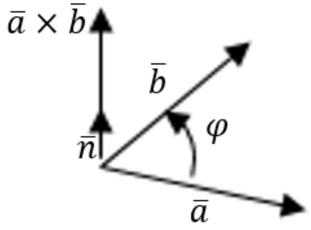


Fig. 3.6.

We defined the vector product $\bar{a} \times \bar{b} = |\bar{a}| \cdot |\bar{b}| \cdot \sin\varphi \cdot \bar{n}$.

The vector $\bar{a} \times \bar{b}$ is orthogonal to both $\bar{a}$ and $\bar{b}$ because it is a scalar multiple of $\bar{n}$.

Since the sines of 0 and π are both zero, it makes sense to define the cross product of two parallel nonzero vectors to be $\bar{0}$.

If one or both of $\bar{a}$ and $\bar{b}$ are zero, we also define $\bar{a} \times \bar{b}$ to be zero.

Nonzero vectors $\bar{a}$ and $\bar{b}$ are parallel if and only if $\bar{a} \times \bar{b} = \bar{0}$.

Reversing the order of the factors inverses the direction of the product. When the fingers of our right hand curl through the angle φ from $\bar{b}$ to $\bar{a}$, our thumb points the opposite way and the unit vector is $-\bar{n}$. Therefore, $\bar{b} \times \bar{a} = -\bar{a} \times \bar{b}$.

The cross product is not commutative.

$\bar{i}\times\bar{j}=\bar{k}$, $\bar{j}\times\bar{i}=-\bar{k}$, $\bar{i}\times\bar{k}=-\bar{j}$, $\bar{k}\times\bar{i}=\bar{j}$, $\bar{j}\times\bar{k}=\bar{i}$, $\bar{k}\times\bar{j}=-\bar{i}$ and

$\bar{i}\times\bar{i}=\bar{j}\times\bar{j}=\bar{k}\times\bar{k}=0$.

$\left|\bar{a}\times\bar{b}\right|$ is the area of a parallelogram.

Because $\left|\bar{n}\right|=1$, $\left|\bar{a}\times\bar{b}\right|=\left|\bar{a}\right|\cdot\left|\bar{b}\right|\cdot\sin\varphi$. $S=\left|\bar{a}\times\bar{b}\right|=\left|\bar{a}\right|\cdot\left|\bar{b}\right|\cdot\sin\varphi$.

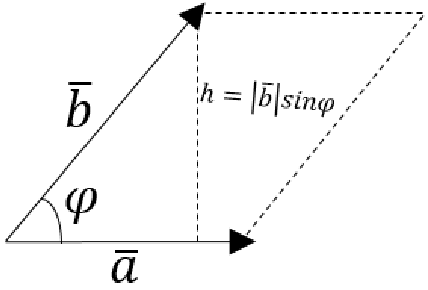


Fig. 3.7. The parallelogram determined by $\bar{a}$ and $\bar{b}$.

Torque.

When we turn a bolt by applying a force $\overline{F}$ to a wrench(fig. 3.8), the torque we produce acts along the axis of the bolt to drive the bolt forward. The magnitude of the torque depends on how far out or the wrench the force is applied and or how much of the force is perpendicular to the wrench at the point of application.

Torque vector $\overline{M}=\bar{r}\times\overline{F}$.

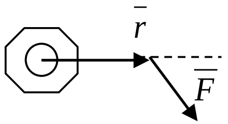


Fig. 3.8. The torque vector describes the tendency of force $\overline{F}$ to drive the bolt forward.

The associative and distributive laws.

As rule, cross-product multiplication is not associative. $\left(\bar{a}\times\bar{b}\right)\times\bar{c}\neq\bar{a}\times\left(\bar{b}\times\bar{c}\right)$.

However, the following laws do hold:

Scalar distributive law

$$(r\bar{a})\times(s\bar{b})=(r\cdot s)\cdot(\overline{a\times b}).$$

Vector distributive laws:

$$\bar{a}\times(\bar{b}+\bar{c})=\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$$

$$(\bar{b}+\bar{c})\times\bar{a}=\bar{b}\times\bar{a}+\bar{c}\times\bar{a}.$$

The determinant formula for $\bar{a}\times\bar{b}$.

Suppose $\bar{a}=x_1\bar{i}+y_1\bar{j}+z_1\bar{k}$ and $\bar{b}=x_2\bar{i}+y_2\bar{j}+z_2\bar{k}$ then

$$\bar{a}\times\bar{b}=x_1x_2\bar{i}\times\bar{i}+x_1y_2\bar{i}\times\bar{j}+x_1z_2\bar{i}\times\bar{k}+y_1x_2\bar{j}\times\bar{i}+y_1y_2\bar{j}\times\bar{j}+y_1z_2\bar{j}\times\bar{k}+z_1x_2\bar{k}\times\bar{i}+$$

$$+z_1y_2\bar{k}\times\bar{j}+z_1z_2\bar{k}\times\bar{k}=x_1y_2\bar{k}-x_1z_2\bar{j}-y_1x_2\bar{k}+y_1z_2\bar{i}+z_1x_2\bar{j}-z_1y_2\bar{i}=$$

$$=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix}.$$

So, if $\bar{a}=x_1\bar{i}+y_1\bar{j}+z_1\bar{k}$, $\bar{b}=x_2\bar{i}+y_2\bar{j}+z_2\bar{k}$ then

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix}.$$

Example. $\bar{a}=2\bar{i}+3\bar{j}-4\bar{k}$, $\bar{b}=3\bar{i}-2\bar{j}+5\bar{k}$. Find $\bar{a}\times\bar{b}$.

Solution. $\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & 3 & -4\\ 3 & -2 & 5\end{vmatrix}=\bar{i}\cdot\begin{vmatrix}3 & -4\\ -2 & 5\end{vmatrix}-\bar{j}\cdot\begin{vmatrix}2 & -4\\ 3 & 5\end{vmatrix}+\bar{k}\cdot\begin{vmatrix}2 & 3\\ 3 & -2\end{vmatrix}=$

$=\bar{i}\cdot(15-8)-\bar{j}\cdot(10-(-12))+\bar{k}\cdot(-4-9)=7\bar{i}-22\bar{j}-13\bar{k}$.

The area of the triangle $A(x_1;y_1;z_1)$, $B(x_2;y_2;z_2)$, $C(x_3;y_3;z_3)$ we can find as $S=\frac{1}{2}\left|\overline{AC}\right|\times\left|\overline{AB}\right|$

$\overline{AC}=(x_3-x_1)\bar{i}+(y_3-y_1)\bar{j}+(z_3-z_1)\bar{k}$

$\overline{AB}=(x_2-x_1)\bar{i}+(y_2-y_1)\bar{j}+(z_2-z_1)\bar{k}$

The triple scalar product.

The product $(\bar{a}\times\bar{b})\cdot\bar{c}$ is called the triple scalar product of $\bar{a}$, $\bar{b}$ and $\bar{c}$ (in that order).

$\left|(\bar{a}\times\bar{b})\cdot\bar{c}\right|=\left|\bar{a}\times\bar{b}\right|\cdot\left|\bar{c}\right|\cdot\cos\varphi$.

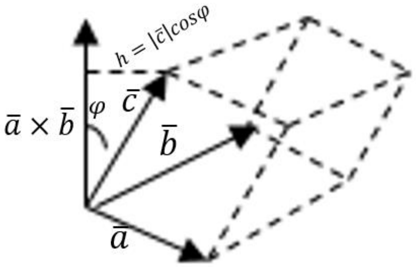


Fig. 3.9. Volume=area of base $\cdot$ height=$\left|\bar{a}\times\bar{b}\right|\cdot\left|\bar{c}\right|\cdot\cos\varphi=\left|(\bar{a}\times\bar{b})\cdot\bar{c}\right|$.

The absolute value of the product is the volume of the parallelepiped, determined by $\bar{a}$, $\bar{b}$ and $\bar{c}$ (fig. 3.9). The number $\left|\bar{a}\times\bar{b}\right|$ is the area of the base parallelogram. The number $|\bar{c}|\cdot\cos\varphi$ is the parallelepiped´s height.

It is easy to see that $\bar{a}\cdot\bar{b}\cdot\bar{c}=\bar{b}\cdot\bar{c}\cdot\bar{a}=\bar{c}\cdot\bar{a}\cdot\bar{b}$.

On the other, hand, $\bar{a}\cdot\bar{b}\cdot\bar{c}=-\bar{c}\cdot\bar{b}\cdot\bar{a}=-\bar{a}\cdot\bar{c}\cdot\bar{b}=-\bar{b}\cdot\bar{a}\cdot\bar{c}$.

The triple scalar product can be evaluated as a determinant.

$$\bar{a}\cdot(\bar{b}\times\bar{c})=\bar{a}\cdot\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix}\bar{i}-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix}\bar{j}+\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}\bar{k}\right)=a_1\cdot\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix}-a_2\cdot\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix}+a_3\cdot\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}=$$

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix}.$$

Therefore, $\bar{a}\cdot(\bar{b}\times\bar{c})=(\bar{a}\times\bar{b})\cdot\bar{c}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix}$.

Example. Find the volume of parallelepiped determined by $\bar{a}=\bar{i}+3\bar{j}-\bar{k}$, $\bar{b}=-3\bar{i}+4\bar{k}$, $\bar{c}=7\bar{j}-4\bar{k}$.

Solution. $\bar{a}\cdot(\bar{b}\times\bar{c})=\begin{vmatrix} 1 & 3 & -1 \\ -3 & 0 & 4 \\ 0 & 7 & -4\end{vmatrix}=0+0+3\cdot7-0-28-36=-43$.

$V=|-43|=43$.

If $\bar{a}\cdot\bar{b}\cdot\bar{c}>0$ then the triple $\bar{a}$, $\bar{b}$, $\bar{c}$ is right-handed. If $\bar{a}\cdot\bar{b}\cdot\bar{c}<0$ then the triple $\bar{a}$, $\bar{b}$, $\bar{c}$ is left-handed.

§4. Lines and planes in space.

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.

Suppose l is a line in space passing through a point $M_0(x_0, y_0, z_0)$ parallel to a vector $\overline{a} = (m; n; p)$. Then l is the set of all points $M(x, y, z)$ for which $\overline{M_0M} = t \cdot \overline{a},\ t \in R$.(fig. 3.1).

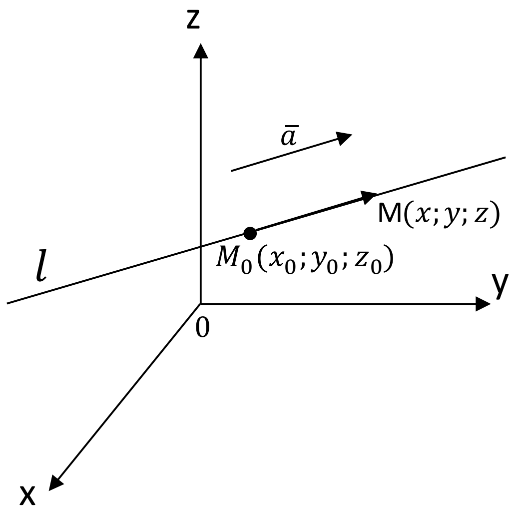


Fig. 4.1. $\overline{a} \parallel \overline{M_0M}$.

Vector equation for the line through $M_0(x_0, y_0, z_0)$ parallel to $\overline{a}$. $\overline{M_0M} = t \cdot \overline{a}, -\infty < t < +\infty$.

Equating the corresponding components of the two sides of this equation gives three scalar equations involving the parameter *t*:

$$(x - x_0)\overline{i} + (y - y_0)\overline{j} + (z - z_0)\overline{k} = t(m\overline{i} + n\overline{j} + p\overline{k})$$

$$\begin{cases} x - x_0 = tm \\ y - y_0 = tn \\ z - z_0 = tp. \end{cases}$$

Standard parametrization of the line through $M_0(x_0, y_0, z_0)$ parallel to $\overline{a} = m\overline{i} + n\overline{j} + p\overline{k}$

$$\begin{cases} x = x_0 + tm \\ y = y_0 + tn \quad t \in R. \\ z = z_0 + tp \end{cases}$$

Example. Find parametric equation for the line through $P(-2;3;-2)$ and $Q(0;-2;4)$.

Solution. The vector $\overline{PQ} = (0-(-2);-2-3;4-\big(-2)) = (2;-5;6)\big)$ is parallel to the line.

Therefore, $\begin{cases} x = -2 + 2t \\ y = 3 + (-5)t \quad t \in R. \\ z = -2 + 6t \end{cases}$

It is parametric equation for the line *PQ*. They simply place you at a different point for a given value of *t*.

To parametrize a line segment joining two points, we first parametrize the line through the points. We then find the *t* – values for the endpoints and restrict *t* to lie in the closed interval bounded by these values. The line equations together with this added restriction parametrize the segment.

The distance from a point to a line in space.

To find the distance from a point *M* to a line that passes through a point M_0 parallel to a vector $\overline{a}$, we find the length of the component of $\overline{M_0M}$ normal to the line (fig. 4.2)

$$d = \left|\overline{M_0M}\right| \cdot \sin\varphi = \frac{\left|\overline{M_0M} \times \overline{a}\right|}{\left|\overline{a}\right|}.$$

M

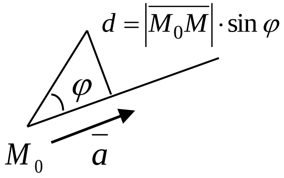


Fig. 4.2. The distance from M to the line through M_0 parallel to $\overline{a}$ is $\left|\overline{M_0M}\right| \cdot \sin\varphi$, where φ is the angle between $\overline{M_0M}$ and $\overline{a}$.

The distance from point M to a line through M_0 parallel to $\overline{a}$: $d = \dfrac{\left|\overline{M_0M} \times \overline{a}\right|}{\left|\overline{a}\right|}$.

Suppose, we have the parametric equations for the line L: $\begin{cases} x = x_0 + tm \\ y = y_0 + tn \quad t \in R. \\ z = z_0 + tp \end{cases}$

We have $t = \dfrac{x - x_0}{m} = \dfrac{y - y_0}{n} = \dfrac{z - z_0}{p}$.

Now we obtain the canonical equation for the line L: $\dfrac{x - x_0}{m} = \dfrac{y - y_0}{n} = \dfrac{z - z_0}{p}$.

Suppose, the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ belong to the line L. In this case, $\overline{a} = \overline{M_1M_2} = (x_2 - x_1; y_2 - y_1; z_2 - z_1)$.

The canonical equations for the line L: $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

The line on the plane.

Consider the line L on the plane xOy. Suppose, the point (0; b) belongs to the line L. Denote by α the angle between the line L and Ox.

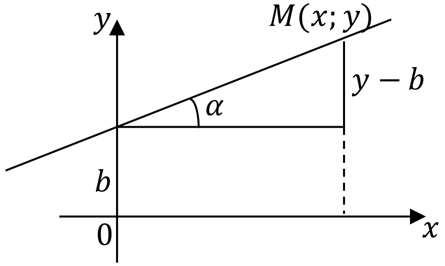


Fig. 4.3. The points (0; *b*) and *M (x; y)* are belong to the line *L*.

Let *M (x; y)* be a point on the line L. In this case, $y-b=x\cdot\operatorname{tg}\alpha$ (fig. 4.3).

Denote $\operatorname{tg}\alpha$ by *k*. We obtain the next equation: $y-b=kx$ or $y=kx+b$.

The angle $\alpha>0$, if we turn the line L shortly from Ox against the direction of hour hand. The number *k* is the angular coefficient of a straight line.

The general equation for the line *L* has the next form: $Ax+By+C=0,\ A,B,C\in R$.

If $B\neq 0$, then $y=-\frac{A}{B}x-\frac{C}{B},\ k=\operatorname{tg}\alpha=-\frac{A}{B},\ b=-\frac{C}{B}$.

Suppose, the point $M_0(x_0,y_0)$ belongs to the line *L* and its angular coefficient is equal k.

In this case, the line *L* has the next equation:

$$y-y_0=k\cdot(x-x_0).$$

Further, let the points $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$ are belong to the line *L*. The line *L* has the next equation:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.$$

If $M_1(a;0)$ and $M_2(0;b)$ are the points of intersection for the line *L* with O*x* and O*y* then the line *L* has the next equation:

$\frac{x}{a}+\frac{y}{b}=1$. (Fig. 4.4). It is the intersect form equation for the line *L*.

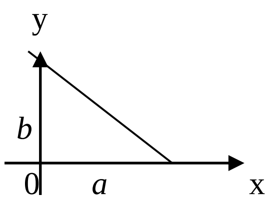


Fig. 4.4. The points (*a*; 0) and (0; *b*) are belong to the line *L*.

Let the point $M_0(x_0, y_0)$ belongs to the line *L* and the vector $\overline{n} = A\overline{i} + B\overline{j}$ is perpendicular to the line *L*.

In this case, the line *L* has the next equation:

$$A \cdot (x - x_0) + B \cdot (y - y_0) = 0 .$$

It is the normal equation for the line *L*.

Equations for planes in space.

Suppose plane π passes through a point $M_0(x_0, y_0, z_0)$ and is normal to the nonzero vector

$\overline{n} = A\overline{i} + B\overline{j} + C\overline{k} = 0$. Then π is the set of all points $M(x, y, z)$ for which $\overline{M_0M}$ is orthogonal to $\overline{n}$. (Fig. 4.5).

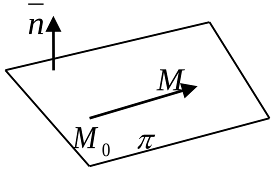


Fig. 4.5. The vector $\overline{M_0M}$ is perpendicular to the vector $\overline{n}$.

That is, M_0 lies on π if and only if $\overline{n} \cdot \overline{M_0M} = 0$.

This equation is equivalent to

$A\cdot(x-x_0)+B\cdot(y-y_0)+C\cdot(z-z_0)=0$.

Plane through $M_0(x_0,y_0,z_0)$ normal to $\bar{n}=A\bar{i}+B\bar{j}+C\bar{k}$. Vector equation $\bar{n}\cdot\overline{M_0M}=0$.

Component equation $A\cdot(x-x_0)+B\cdot(y-y_0)+C\cdot(z-z_0)=0$.

Example. Find an equation for the plane through $M_0(-5;0;4)$ perpendicular to $\bar{n}=3\bar{i}+2\bar{j}-\bar{k}$.

$3\cdot(x-(-5))+2\cdot(y-0)+(-1)\cdot(z-4)=0$.

$3\cdot(x+5)+2y-z+4=0$

$3x+15+2y-z+4=0$ or $3x+2y-z+19=0$.

A plane determined by three points.

Suppose the points $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ and $M_3(x_3,y_3,z_3)$ are belong to the plane π.

In this case, the triple product $\overline{M_1M}\cdot\overline{M_1M_2}\cdot\overline{M_1M_3}$ is equal 0, for any point $M\in\pi$.

Therefore, $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$.

The distance from the point $M_0(x_0,y_0,z_0)$ to the plane π $(Ax+By+Cz+D=0)$ is determined by the next formula: $d=\dfrac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$.

Angles between planes; lines of intersection.

The angle between two intersecting planes is defined to be the(acute) angle determined by their normal vectors: (fig. 4.6).

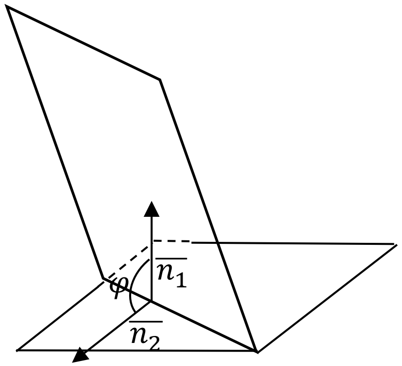


Fig. 4.6. The angle between two planes is obtained from the angle between their normals.

Example. Find the angle between the planes $3x-6y+5z-15=0$ and $2x+y-2z-5=0$.

Solution. The vectors $\overline{n_1}=(3;-6;5)$ and $\overline{n_2}=(2;1;-2)$ are normals to the planes.

$$\cos\varphi=\frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\cdot|\overline{n_2}|}=\frac{6-6-10}{\sqrt{9+36+25}\cdot\sqrt{4+1+4}}=\frac{-10}{\sqrt{70}\cdot\sqrt{9}}=-\frac{10}{3\sqrt{70}}.$$

We have $\cos\varphi<0$, therefore $\varphi\in(90^\circ;180^\circ)$. In this case, $\cos\alpha=|\cos\varphi|=\frac{10}{3\sqrt{70}}$.

Here α denotes the angle between two planes, φ is the angle between two vectors $\overline{n_1}$ and $\overline{n_2}$.

$$\cos\alpha=\frac{10}{3\sqrt{70}}=\frac{10}{3\cdot 8{,}37}=0{,}40$$

$$\alpha=\arccos(0{,}40)=66{,}4^\circ.$$

The intersection L of two planes π_1 $(Ax_1+By_1+Cz_1+D_1=0)$ and π_2 $(Ax_2+By_2+Cz_2+D_2=0)$ has next equations: $L:\begin{cases}Ax_1+By_1+Cz_1+D_1=0\\ Ax_2+By_2+Cz_2+D_2=0\end{cases}.$

It is the general equations for the line L.

Example. Find parametric equations for the line in which the planes $2x-5y-z-15=0$ (π_1) and $3x+2y+z-1=0$ (π_2) intersect.

Solution. We find a vector parallel to the line and a point on the line and use the canonical equations for the line L.

$\overline{n_1}=(2;-5;-1)$, $\overline{n_2}=(3;2;1)$. $\overline{a}=\overline{n_1}\times\overline{n_2}$. In this case $\overline{a}\parallel\pi_1$ and $\overline{a}\parallel\pi_2$. Therefore, $\overline{a}\parallel L$.

$$\overline{a}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 2 & -5 & -1\\ 3 & 2 & 1\end{vmatrix}=\overline{i}\cdot(-5+2)-\overline{j}\cdot(2+3)+\overline{k}\cdot(4+15)=-3\overline{i}-5\overline{j}+19\overline{k}.$$

On the other hand, let us find a point $M_0(x_0,y_0,z_0)\in L$.

In this case, $M_0\in\pi_1$ and $M_0\in\pi_2$:

$\begin{cases}2x_0-5y_0-z_0-15=0\\ 3x_0+2y_0+z_0-1=0\end{cases}$. There are infinitely many solutions for the system. Suppose, z_0 is equal 0.

$$\begin{cases}2x_0-5y_0-15=0 & \cdot(-3)\\ 3x_0+2y_0-1=0 & \cdot 2\end{cases}\qquad \left.\begin{cases}-6x_0+15y_0+45=0\\ 6x_0+4y_0-2=0\end{cases}\quad\right]+$$

$$19y_0+43=0,\ y_0=-\frac{43}{19},\qquad 2x_0=5y_0+15,\ \ 2x_0=-\frac{215}{19}+15=\frac{70}{19},\ \ x_0=\frac{35}{19}.$$

$$M_0\left(\frac{35}{19};-\frac{43}{19};0\right)\in L.$$

Therefore, the line L has the next canonical equations: $\dfrac{x-\frac{35}{19}}{-3}=\dfrac{y+\frac{43}{19}}{-5}=\dfrac{z-0}{19}$.

§5. Quadratic Curves.

We assume that a rectangular Cartesian coordinate system *Oxy* is given on the plane.

An ***equation of the second degree*** with two variables is an equation of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad A^2 + B^2 + C^2 \neq 0. \quad (5.1)$$

Line (curve) of the second order is the set of point on the plane, the coordinates of which satisfy any equations of the second degree (5.1).

Lines of the second order are ***a circle***, ***an ellipse***, ***a hyperbola***, ***a parabola***. Let as consider the equations of these lines in the simplest (canonical) form, which is achieved by a certain choice of the coordinate system.

5.1 Circles.

Definitions. ***A circle*** is a set of points in a plane that are equidistant from a fixed point $N(a,b)$. The fixed point is called the ***center.*** The line segment that joins the center with any point of the circle is called the ***radius R***. (fig. 5.1).

The canonical equations of the circle.

$$(x-a)^2 + (y-b)^2 = R^2. \quad (5.2)$$

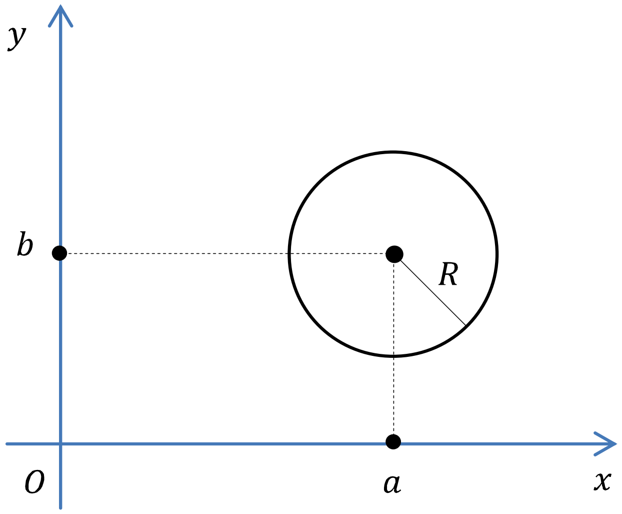


Figure 5.1.

Equation of a circle centered at the origin (5.3) is known as the canonical equation of the circle.

$$x^2 + y^2 = R^2. \tag{5.3}$$

If t is a real parameter, then $\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$

are parametric equations of the circle. By elimination the parameter t, we return to the canonical equation.

5.2 Ellipses

Definitions. ***An ellipse*** is the set of points on the plane the sum of whose distances from the two fixed points is a constant. These two fixed points F_1 and $F_2, |F_1F_2| = 2c$ are called the ***foci*** (plural of ***focus***).

One of Kepler's laws is that the orbits of a planet in the solar system are ellipses with the sun at one focus. In order to obtain the simplest equation for an ellipse, we place the foci on the Ox -axis at the points $F_1(c;0)$, $F_2(-c;0)$, so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0\,(2a > 2c)$, we obtain the canonical equation of the ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (5.4)$$

In this equation the numbers $a > b; b > 0$ and a, b, c are related as follows:

$$c^2 = a^2 - b^2. \qquad (5.5)$$

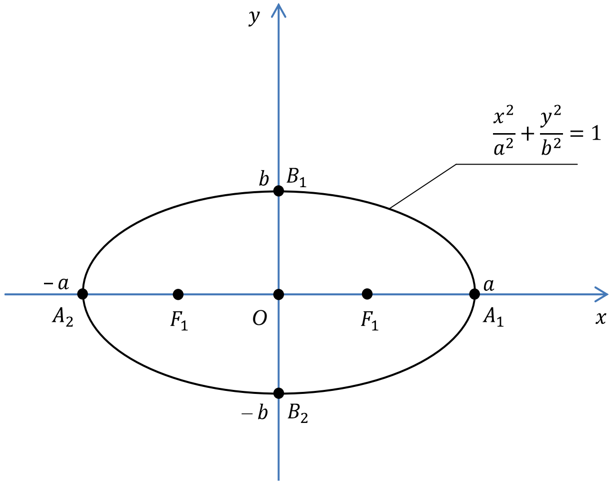


Figure 5.2.

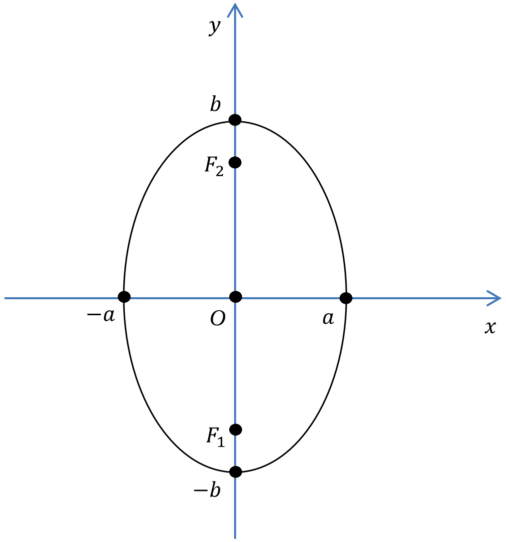


Figure 5.3.

The line segment joining the vertices (-*a*;0) and (*a*; 0) is called the ***major axis***. Numbers ***a*** and ***b*** are called the semi-axes of the ellipse, ***a*** – ***major semi-axis***, *b* – ***minor semi-axis***.

The corresponding points $A_1(a;0); A_2(-a;0); B_1(0;b); B_2(-b;0)$ are called the vertices of the ellipse (these are the intersection points of the ellipse with the coordinate axes), the point $O(0;0)$ is ***the center*** of the ellipse. The ellipse given by equation (5.4) is shown in fig. 5.2.

The eccentricity of an ellipse is the number ε (epsilon), equals to the ratio of the distance between ellipse foci 2*c* to the length of the major axis 2*a*:

$$\varepsilon = \frac{c}{a} \quad (0 \le \varepsilon < 1\text{, because } c < a). \qquad (5.6)$$

Remarks. 1) For $a<b$ equation (5.4) also defines an ellipse, but its foci lie on the *Oy* axes, while $c^2 = b^2 - a^2; F_1(0;-c);\ F_2(0;c); \varepsilon = \frac{c}{b}$ (fig. 5.3).

2) If $a=b$, then equation (5.4) defines a circle of radius *a* centered at the origin: $x^2 + y^2 = a^2$. In this case, $c=0$. Therefore, a circle is a special case of an ellipse with coinciding foci, with $\varepsilon = 0$.

3) The equation of an ellipse with axes parallel to the coordinate ones has the form:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1, \qquad (5.7)$$

where x_0, y_0 – are the coordinates of the center of the ellipse.

4) Equations: $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$, $t \in [0;2\pi]$, are the parametric equations of the ellipse.

5.3 Hyperbola

Definitions. ***A hyperbola*** is the set of all points in a plane, modulus the distances from each of which to two given points, called ***foci***, is a constant value, less than the distance between the foci, and different from zero.

If the focuses of the hyperbola F_1 and $F_2, |F_1F_2| = 2c$, locate on the Ox axes symmetrically with respect to the origin: $F_1(c;0)$, $F_2(-c;0)$; and denote the modulus of the difference from the point of the curve to the foci through $2a(0 < a < c)$, then the canonical equation of the hyperbola looks like:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad (5.8)$$

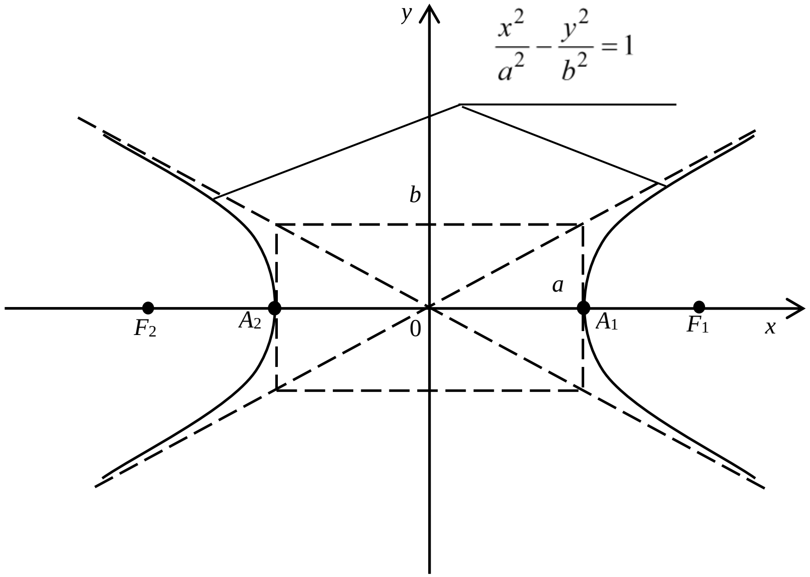
where $b>0$ and $c^2 = a^2 + b^2$. (5.9)

The hyperbola (5.8) intersects the *Ox* axes at the points $A_1(a;0); A_2(-a;0)$, which are called the ***vertices*** of the hyperbola, and the hyperbola does not intersect the *Oy* axes. Any hyperbola has two vertices.

The axis of the hyperbola that it intersects is called the ***real*** axis, and the one that does not intersect is the ***imaginary*** axes. The numbers *a* and *b* are called ***semi-axes*** of the hyperbola; (*a* –***real***, *b* – ***imaginary*** semi-axes). The point $O(0;0)$ is called the ***center*** of the hyperbola.

Two intersecting lines $y = \pm\frac{b}{a}x$ are the ***asymptotes*** of the hyperbola. With unlimited removal from the origin, the hyperbola approaches its asymptotes infinitely close without crossing it. The hyperbola (5.8) is shown in fig. 5.4.

Figure 5.4.



The eccenyricity of the hyperbola is the number ε (epsilon), equal to the ratio of half the distance between the foci of the hyperbola to its real semi-axes:

$$\varepsilon = \frac{c}{a} \quad (\varepsilon > 1\text{, because } c > a). \qquad (5.10)$$

Remarks. 1) If the foci of the hyperbola lie on the *Oy* axes, then the equation of the hyperbola has the next form:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 . \qquad (5.11)$$

The real axis of the hyperbola (5.11) is the *Oy*-axes, and the imaginary axis is the *Ox*-axes. Hyperbolas given by equation (5.8) and (5.11) are called ***conjugate***. Conjugate hyperbolas have the same asymptotes, and the real axes are perpendicular. For hyperbola (5.8) $\varepsilon = \frac{c}{b}$. It is shown in the fig. 5.5.

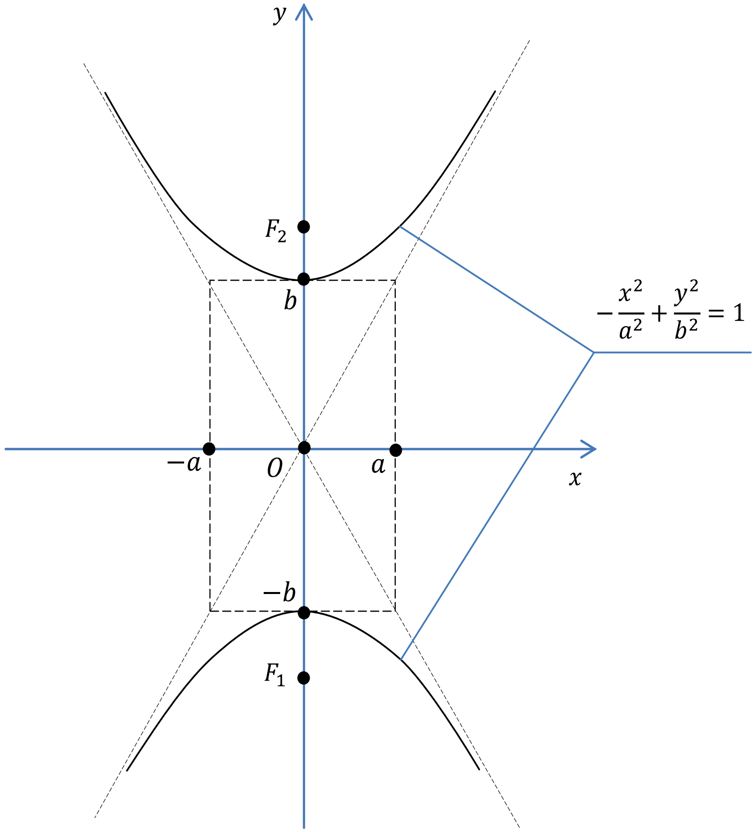


Figure 5.5

2) The equation of a hyperbola with axes parallel to the coordinate ones has the form:

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1, \qquad (5.12)$$

where (x_0, y_0) – are the coordinates of the center of the hyperbola.

5.4 Parabolas

Definitions. ***A parabola*** is the locus of points, which are equidistant from a given point ***F*** and a line ***L***. The point *F* is called the f***ocus***. The line *L* is called the ***directrix*** of the parabola.

If the distance from the focus *F* to the directrix *L* is denoted by ***p***, and the focus *F* is placed on the Ox-axis, which is perpendicular to the directrix *L*, and the Oy-axis is in the middle between the focus and the directrix parallel to the Oy, then the parabola is given by the equation:

$$y^2 = 2px, \qquad (5.13)$$

The equation is called the canonical equation of the parabola (Figure 5.6).

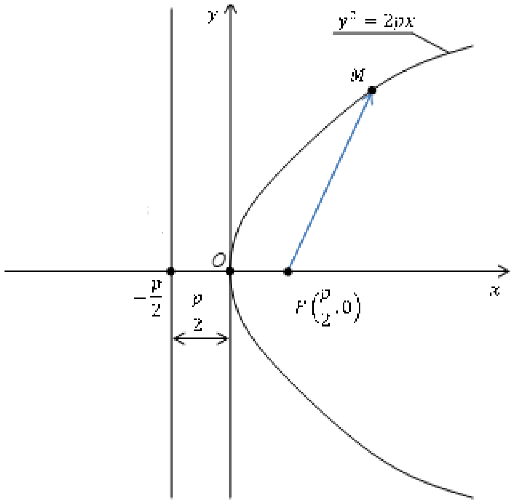


Figure 5.6

The number ***p*** equal to the distance from the foci F to the directrix L, is called the ***parameter*** of the parabola, the point $O(0;0)$ is its ***vertex***, and the Ox-axes is the ***axis of symmetry*** of the parabola. ***The directrix equations L*** is: $x = -\frac{p}{2}$. ***Eccenyricity*** for ***parabolas*** by definition is considered to be equal to 1: $\varepsilon = 1$.

Remarks. 1) The equation $y^2 = -2px$ also specifies a parabola that is symmetric with respect to the Ox-axes (fig. 5.6). The focus F has coordinates $F\left(-\frac{p}{2};0\right)$, and the equation for directrix has the next form: $x = \frac{p}{2}$.

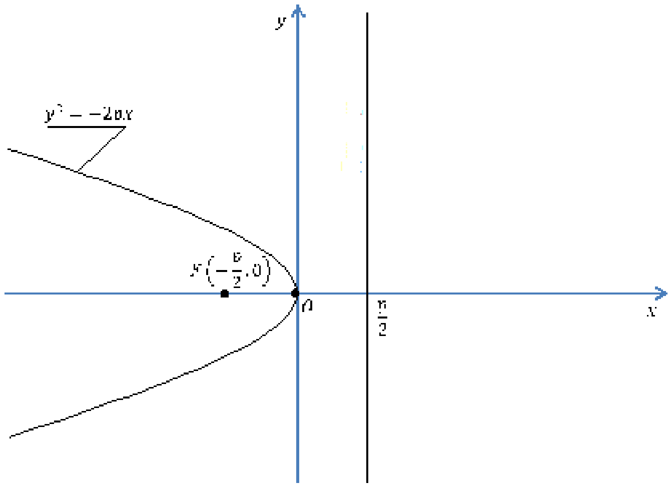


Fig 5.7. directrix.

2) The equations $x^2 = 2py; x^2 = -2py$ also specify p abolas, but are symmetrical with respect to the *Oy*-axes. The parabola $x^2 = 2py$ has a focus $F\left(0;\frac{p}{2}\right)$, the equation of the directrix $y = -\frac{p}{2}$ (fig.5.8). For a parabola $x^2 = -2py$ the focus $F\left(0;-\frac{p}{2}\right)$, and the equation for the directrix $y = \frac{p}{2}$ (fig. 5.9).

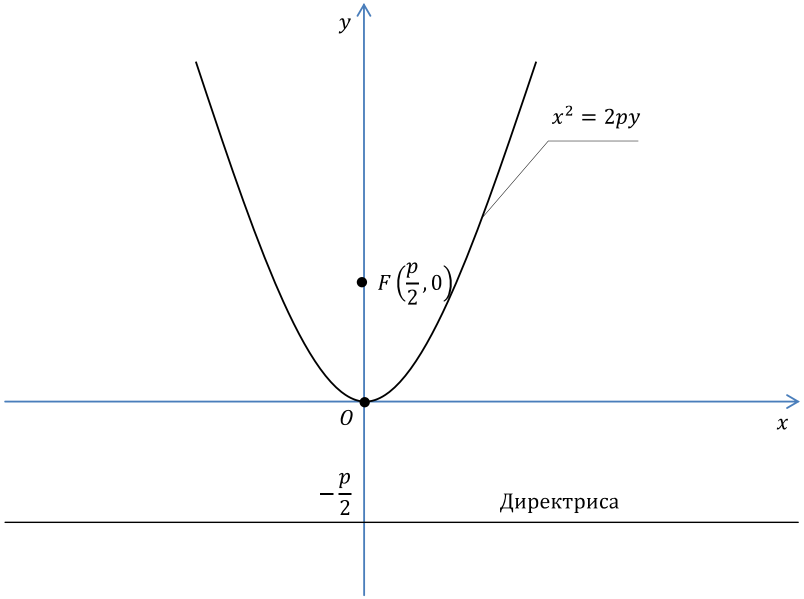
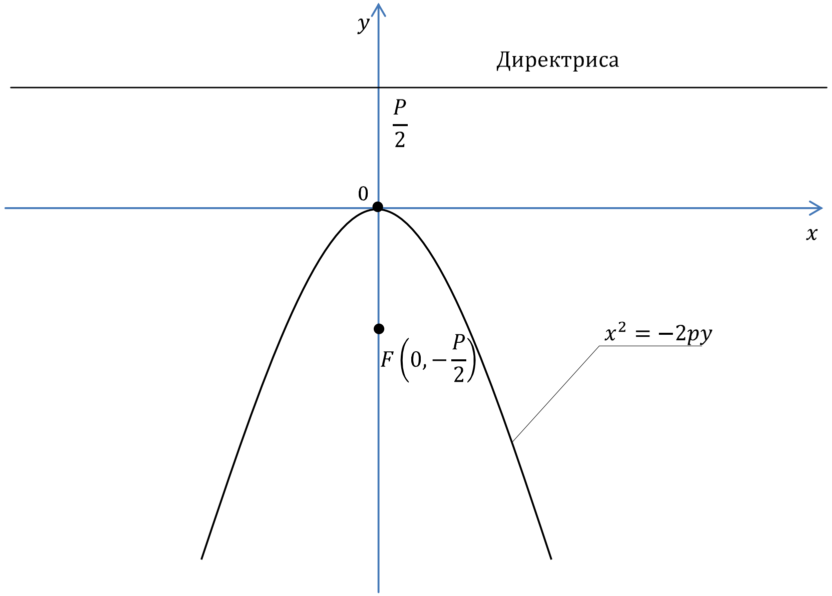


Fig. 5.8.

Fig. 5.9.



3) Parabola equations with symmetry axes parallel to coordinate axes have the form: $(y-y_0)^2=\pm 2p(x-x_0)$; $(x-x_0)^2=\pm 2p(y-y_0)$.

It is known that for any line of the second order on the plane there is a rectangular Cartesian coordinate system in which this line is given by the canonical equation.

Let us show with specific examples how to practically bring an equation of a second order line that it does not contain a term with a product of variables, $B=0$ in (4.77), to the canonical form.

Example. The second-order line is given by the equation.

$3y^2-12x-6y+11=0$.

Reduce the equation to the canonical form, give the detailed description of the curve and draw it.

Solution. This equation is not canonical. Let's select the full square, which includes all the terms with the variable y, ***and the coefficient at y^2 must***

be taken out of brackets.

$$3\left(y^2-2y\right)-12x+11=0 \Leftrightarrow 3\left(\left(y^2-2y+1\right)-1\right)-12x+11=0 \Leftrightarrow$$

$$3(y-1)^2-3-12x+11=0 \Leftrightarrow 3(y-1)^2=12x-8 \Leftrightarrow (y-1)^2=4\left(x-\frac{2}{3}\right).$$

Applying the parallel transfer transform $X=x-\frac{2}{3};\ Y=y-1$, from the last equation we obtain the canonical equation $Y^2=4X$. From here we can see that the line in question is a parabola symmetrical with respect to the O_1X-axes.

To draw this line, let's draw both the *Oxy* and O_1XY coordinate systems in one picture. With parallel transfer, the coordinate axes move parallel to themselves, so to determine their location, it is enough to determine the position of the new origin.

At the point O_1 : $X=0$ and $Y=0$, then, $x=\frac{2}{3};\ y=1$. Through the point $O_1\left(\frac{2}{3};1\right)$ we draw axes co-directed to the axes *Ox* и *Oy*, and we get a new coordinate system. In this system, we draw a parabola $Y^2=4X$, whose vertex is at the origin, and the branches are directed towards the positive direction of the O_1X symmetrically to this axis (fig. 5.10).

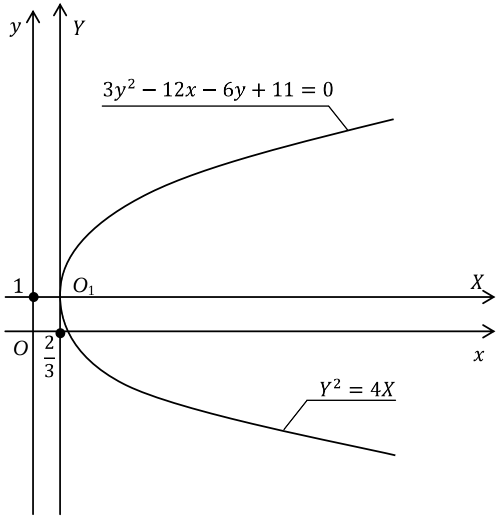


Fig. 5.10.

Example. Simplify the equation $2x^2+5y^2-12x+10y+13=0$ by using origin translation. Construct a line defined by this equation.

Solution. Let’s complete the full squares by the variables x and y, respectively.

$$2\left(x^2-6x\right)+5\left(y^2+2y\right)+13=0 \Leftrightarrow$$

$$2\left(x^2-6x+9\right)-18+5\left(y^2+2y+1\right)-5+13=0 \Leftrightarrow$$

$$2(x-3)^2+5(y+1)^2=10 \Leftrightarrow$$

$$\frac{(x-3)^2}{5}+\frac{(y+1)^2}{2}=1.$$

Denote by $x-3=X$, $y+1=Y$, we get the canonical equation of the ellipse $\frac{X^2}{5}+\frac{Y^2}{2}=1.$ The origin of a new coordinate system is the point $O_1(3,-1)$; the O_1X, O_1Y axes are parallel to the *Ox* and *Oy* axes respectively. Semi-major axis of the ellipse $a=\sqrt{5}$, semi-minor axis $b=\sqrt{2}.$ Let's draw a curve in fig. 5.11.

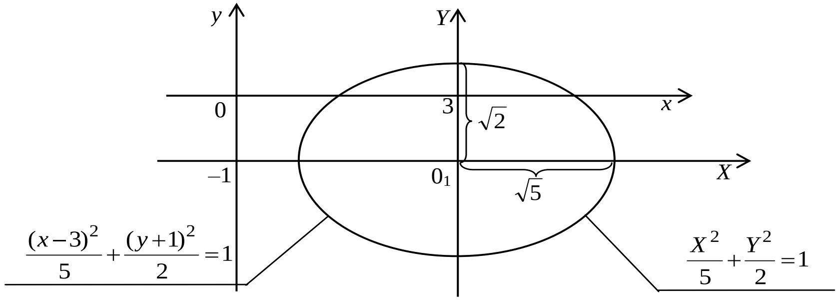


Fig. 5.11.

Example. Complete an equation for a circle that has a center at a point $N(2;-5)$ and a radius of 4.

Solution. Let's substitute the values of the coordinates of the center and radius into the equation (5.2), we get $(x-2)^2+(y+5)^2=16$.

Example. The ellipse equation is given $24x^2+49y^2=1176$. Find: 1) the length of its semi-axes; 2) focal points; 3) eccentricity.

Solution. Reduce the ellipse equation $24x^2+49y^2=1176$ to the canonical form (4.80), dividing both parts of the equation by 1176: $\frac{x^2}{49}+\frac{y^2}{24}=1$, from which the following relations follow:

$a^2=49, b^2=24$, т. е. $a=7$ – semi-major axis; $b=2\sqrt{6}$ – semi-minor axis.

Using the equality (5.5), let's find $c^2=a^2-b^2=49-24=25, c=5$. So, $F_1(5;0), F_2(-5;0)$.

According to the formula (5.6) $\varepsilon=\frac{c}{a}=\frac{5}{7}$.

Example. Complete an equation for an ellipse passing through the points $M_1(2;-4\sqrt{3}), M_2(-1;2\sqrt{15})$.

Solution. We look for the ellipse equation as: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Since the ellipse passes through the points M_1, M_2, their coordinates satisfy the ellipse equation: $\begin{cases}\frac{4}{a^2}+\frac{48}{b^2}=1,\\ \frac{1}{a^2}+\frac{60}{b^2}=1.\end{cases}$

Multiplying the second equality by (−4) and adding with the first, we find: $-\frac{192}{b^2}=-3$, i.e. $b^2=64$. Substituting the resulting value b^2 into the second equality, we get $\frac{1}{a^2}+\frac{60}{64}=1$, from where $a^2=16$. The ellipse equation sought is as follows: $\frac{x^2}{16}+\frac{y^2}{64}=1$.

Example. The hyperbola equation is given $16x^2-9y^2=144$. Find: 1) the length of the hyperbola semi-axes; 2) focal points; 3) eccentricity; 4) asymptote equations.

Solution. Divide both parts of the equation by 144, there by bringing it to the canonical form (5.8):

$$\frac{x^2}{9}-\frac{y^2}{16}=1.$$

1) From the last equation $a^2=9, b^2=16$, i. e. $a=3$ – the real semi-axes, $b=4$ – the imaginary semi-axis.

2) Using the equality (4.85): $c^2=a^2+b^2$, we get: $c^2=25$, $c=5$. So, $F_1(5;0), F_2(-5;0)$.

3) According to the formula (4.86) $\varepsilon=\frac{c}{a}=\frac{5}{3}$.

4) The asymptote equations are: $y=\pm\frac{b}{a}x$. In our case $y=\pm\frac{4}{3}x$.

Example. Complete an equation for a hyperbola if its foci lie on the *Oy* axis and the distance between them is 10, and the length of the real axis is 8.

Solution. The equation sought is: $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$. According to the condition $2b=8, 2c=10$. So, $b=4, c=5$. From (5.9) $c^2=a^2+b^2$. Let's find the imaginary semi-axis a: $a^2=c^2-b^2=5^2-4^2=9$, т. e. $a=3$. The desired hyperbola equation is: $\frac{y^2}{16}-\frac{x^2}{9}=1$.

Example. The parabola is given by the equation $x^2=4y$. Find: 1) the focus coordinates; 2) the directrix equation.

Solution. The parabola is given by the canonical equation $x^2=2py$. Therefore, $2p=4, p=2$ It is follows from here: 1) $F\left(0;\frac{p}{2}\right)\Rightarrow F(0;1)$; 2) $y=-\frac{p}{2}\Rightarrow y=-1$ – is the equation of the directrix.

Example. Complete an equation for a parabola if its vertex coincides with the origin and the focus is at $F(2;0)$.

Solution. Since the focus of the parabola is always on its axis of symmetry, in our case the axis of the parabola will be the *Ox*-axes, and $F\left(\frac{p}{2};0\right)\Rightarrow F(2;0)$. Hence $\frac{p}{2}=2$, then, $p=4$. In this case, the canonical equation of the parabola would be $y^2=2px$, i. e. in this case $y^2=8x$. Note that the equation for the directrix has the next form: $x=-\frac{p}{2}$, i. e. $x=-2$.

§6 Second-order surfaces

In this section we will consider only the rectangular coordinate system *Oxyz.*

6.1 Cylinders and cones

Let some line L and a straight line α intersecting L be given in space.

Definition. A ***cylindrical surface*** or ***cylinder*** is a surface formed by a straight line α when it moves parallel to itself so that it crosses the line L all the time. The straight lines obtained by moving the straight line α are called the ***generators*** of this cylinder, and the line L is its ***guide***. If the cylinder has an axis of symmetry, and the guide lying in a plane perpendicular to this axis is a circle, then the cylinder is called ***circular***.

Note that if one of the coordinates is missing in the equation of the surface, then this equation defines a cylinder with generators parallel to the coordinate axis defined by the missing coordinate. The equation of the guide of this cylinder lying in the coordinate plane perpendicular to the generators coincides with the equation of the cylinder itself.

So, the equation $F(x, y) = 0$ defines a cylinder with generators parallel to the *Oz* axis. If this equation is considered in the *Oxy* coordinate system (i.e. in the plane $z = 0$), then we get the equation of the guide for this cylinder.

The equations $F(x, z) = 0$ and $F(y, z) = 0$ define cylinders with generators parallel to the axes *Oy* and *Ox*, respectively.

Let some line L and a point O' be given in space.

Definition. A ***conical surface*** or ***cone*** is a surface formed by straight lines passing through O' and intersecting the curve L. The lines that make up the cone are called its ***generators***, and the point O' – is its ***vertex.*** If a cone has an axis of symmetry, and all its generators are inclined to it at the same angle, then the cone is called ***circular.***

6.2 Canonical equations of second-order surfaces

Definition. A ***surface of the second order*** is the set of all points of space satisfying in some coordinate system some equation of the second degree, i.e. an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Kz + R = 0 \quad .(6.1)$$

Equation (6.1) is called the ***general equation*** of the second-order surface (coefficients A, B, C, D, E, F are not equal to zero at the same time).

For any surface of the second order in space, there is a rectangular Cartesian coordinate system in which this surface is given by the canonical equation. We list all fundamentally possible types of canonical equations of the second degree with three variables and thus classify surfaces of the second order.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ – ellipsoid (fig. 6.1);

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ – imaginary ellipsoid;

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ – point O(0,0,0);

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ – single-cavity hyperboloid (fig. 6.2);

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ – double-cavity hyperboloid (fig. 6.3);

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ – a cone of the second order (fig. 6.4);

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$ – elliptical paraboloid (fig. 6.5);

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ – elliptical cylinder (fig. 6.6);

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$ – imaginary elliptical cylinder;

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ – straight line (*Oz* axis);

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$ – hyperbolic paraboloid (fig. 6.7);

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ – hyperbolic cylinder (fig. 6.8);

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ or $\left[\begin{array}{l}\frac{x}{a}-\frac{y}{b}=0\\ \frac{x}{a}+\frac{y}{b}=0\end{array}\right.$ – a pair of intersecting planes;

$y^2=2px$ – parabolic cylinder (fig. 6.9);

$y^2=a^2$ or $\left[\begin{array}{l}y=a\\ y=-a\end{array}\right.$ – a pair of parallel planes;

$y^2=0$ – double plane;

$y^2=-a^2$ – a pair of imaginary parallel planes.

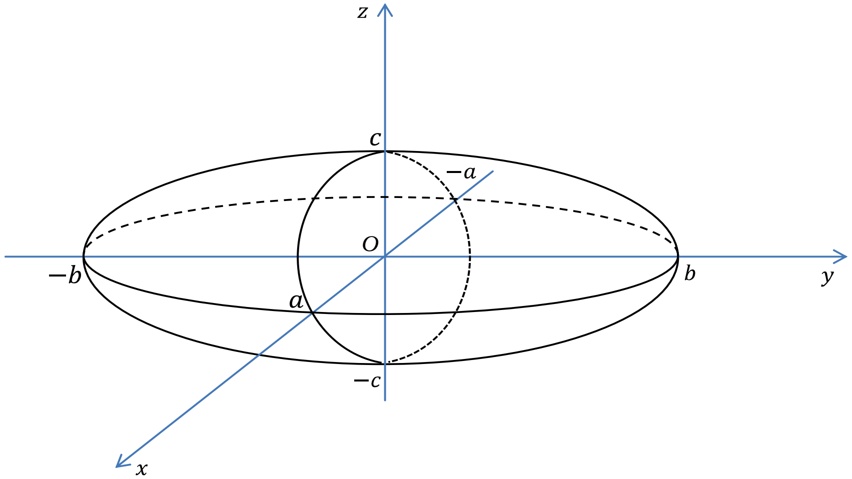


Fig. 6.1.

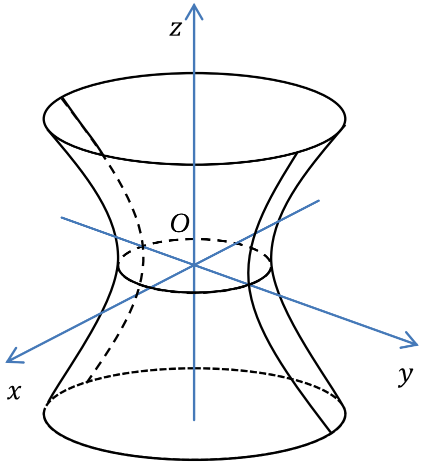


Fig. 6.2.

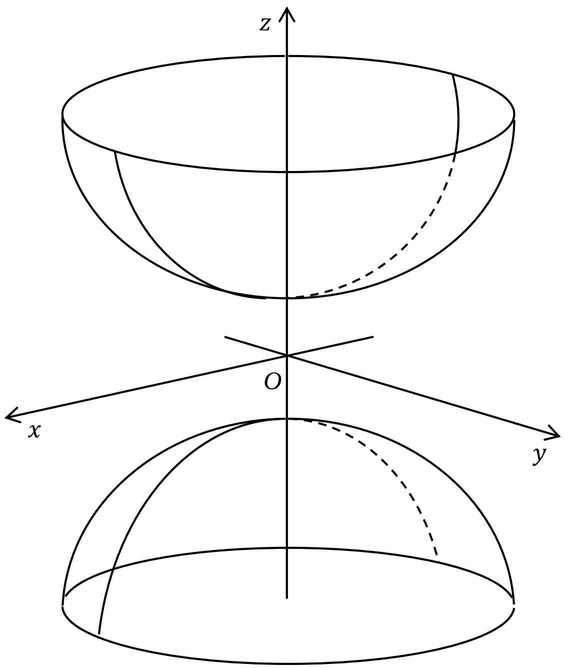


Fig. 6.3.

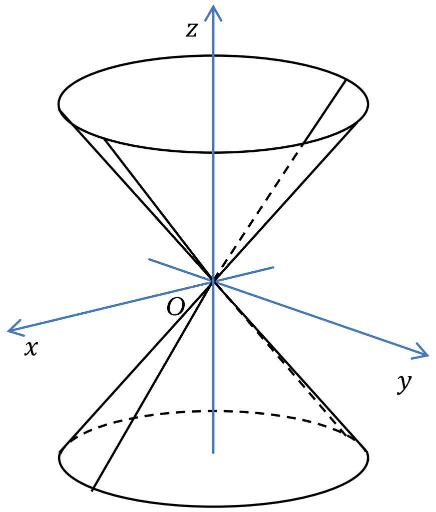


Fig. 6.4.

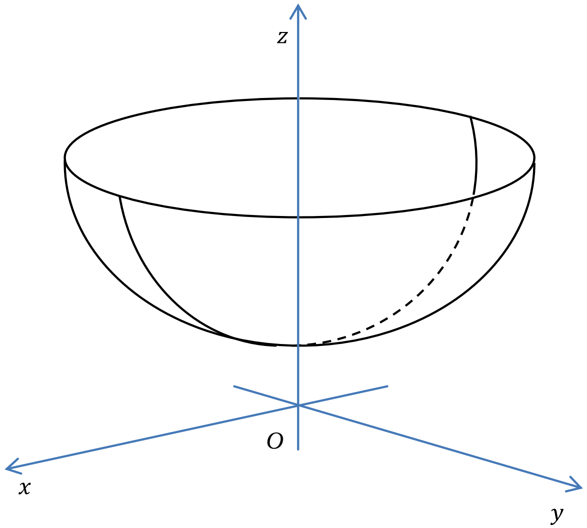


Fig. 6.5.

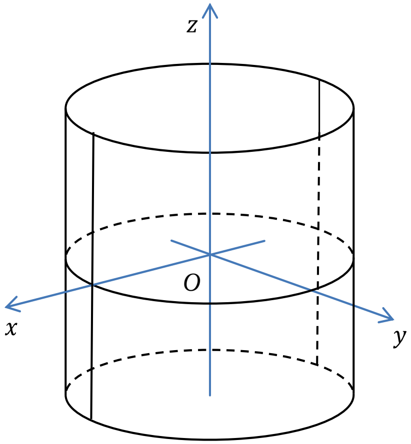


Fig. 6.6.

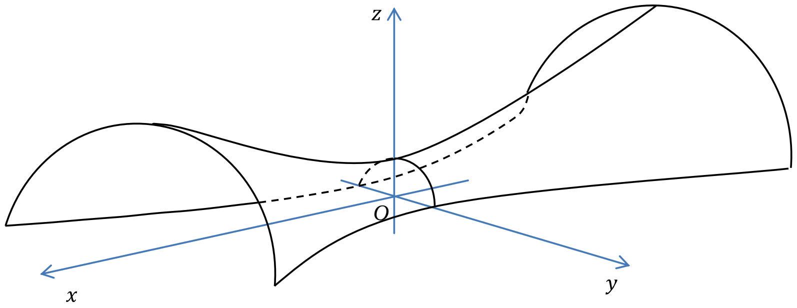


Fig. 6.7.

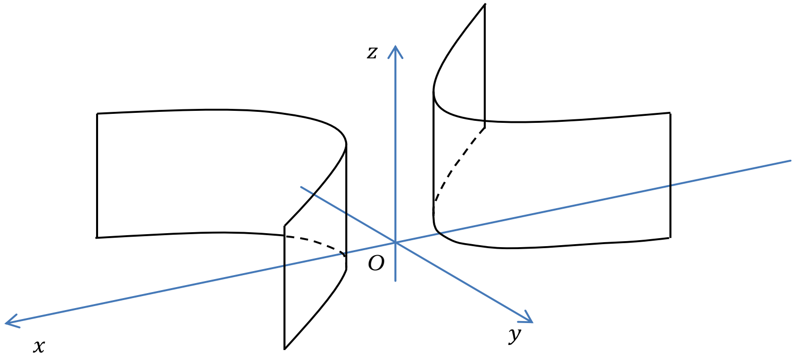


Fig. 6.8.

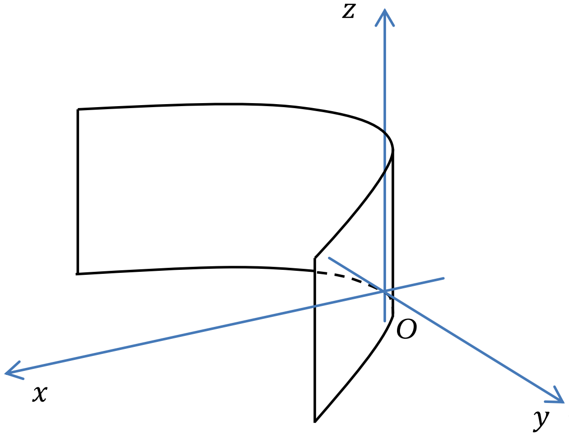


Fig. 6.9.

The numbers ***a, b,*** and ***c*** in the equations of an ellipsoid, two-cavity and one-cavity hyperboloids, a cone of the second order, elliptic and hyperbolic cylinders are called their ***semi-axes***. All these surfaces are symmetric with respect to all coordinate planes and with respect to the origin. The intersection points of a two-cavity hyperboloid and an elliptical paraboloid with axes of symmetry are called their ***vertices***. The ***vertex of a cone*** is its center of symmetry.

In order to determine the type of surface using the canonical equation without formally memorizing the equations, it is useful to reason as follows.

1) If the canonical equation does not contain one of the variables, then it is one of the cylinders. At the same time, its generators are parallel to the coordinate axis determined by the missing variable, and the equation of the guide lying in the coordinate plane perpendicular to this axis coincides with the equation of the surface itself.

2) If the canonical equation contains all the variables, check if they are all squared. If there is a summand of the first degree, then it is one of the paraboloids. Which one of them is easy to understand on the left side of the equation.

3) If there are squares of all three variables in the canonical equation, let's look at the signs of the coefficients for squares. In the case when all these coefficients are of the same sign, we have either an ellipsoid, real or imaginary, or a point. When $a=b=c$ using the ellipsoid equation, we obtain a special case – the equation of the sphere.

4) If the squares of all three variables are present in the canonical equation, but they are of different signs, then the surface under consideration is either one of the hyperboloids or a cone of the second order. The cone is distinguished by the fact that it passes through the origin, which is easily checked by substitution. Since there are only three variables, then two coefficients with squares will have the same sign, and the third will have the opposite sign. Thus, two variables will be equal, and the third one will be special. A cone or hyperboloid is always "elongated" along an axis defined by a special variable. To distinguish a one-cavity hyperboloid from a two-cavity hyperboloid, this special variable should be set equal to zero. If an ellipse is obtained in the section, then the hyperboloid is single-cavity. If the empty set is obtained than the surface is double-cavity.

Example. Construct a body bounded by surfaces

$$x^2+y^2+z^2=4, \;\; x^2+y^2=3z.$$

Solution. The body is bounded from below by the surface of the paraboloid: $x^2+y^2=3z$, and from above by the surface of the sphere: $x^2+y^2+z^2=4$.

The body is shown in fig. 6.10.

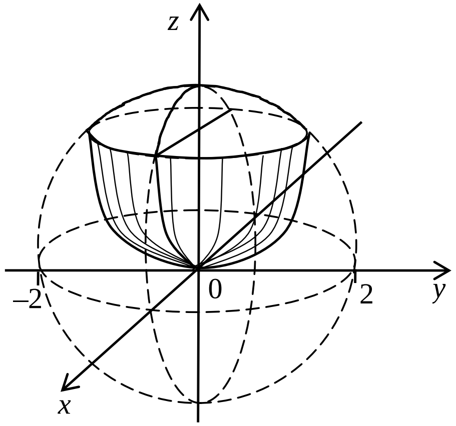


Fig. 6.10.

Example. Construct a body bounded by the surfaces

$$\frac{x^2}{4}+\frac{z^2}{9}=1, \quad x=0, z=0 \quad (x\geq 0, z\geq 0), \quad y=3, y=-3.$$

Solution. The surface $\frac{x^2}{4}+\frac{y^2}{9}=1$ – is an elliptical cylinder. It is intersected by the planes x = 0, z = 0 (coordinate planes Ozy and Oxy). Along the Oy axis the body is bounded by planes y = 3, y = –3 (fig. 6.11).

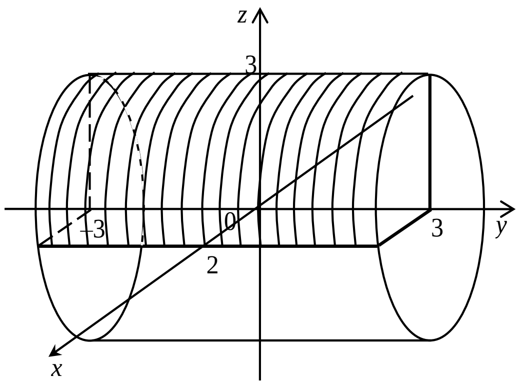


Fig. 6.11.

§7. Sequences

7.1 A numerical sequence.

A numerical sequence is understood as a function

$$x_n = f(n) \qquad (7.1)\ ,$$

defined on the set of natural numbers *N*. Briefly, the sequence is denoted as $\{x_n\}$ or $x_n,\, n \in N$. The number x_1 is called the first term of the sequence. The number x_2 is called the second term of the sequence. The number x_n is called the ***general term*** or ***variable*** of the sequence.

Most often the sequence is given by the ***general term formula***. Formula (7.1) allows you to calculate any member of the sequence by the number n, from it you can immediately calculate any member of the sequence.

So, the equalities $v_n = n^2 + 1,\ z_n = (-1)^n n,\ y_n = \frac{1}{n},\ u_n = \frac{n-1}{n},\ n \in N$ define, respectively, the sequences:

$$v_n = \{2,5,10,\ldots,\cdot n^2+1,\ldots\}$$

$$z_n = \{-1,2,-3,4,\ldots,(-1)^n \cdot n,\ldots\}$$

$$y_n = \left\{1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\right\}.$$

$$u_n = \left\{0,\frac{1}{2},\frac{2}{3},\frac{3}{4}\ldots,\frac{n-1}{n},\ldots\right\}.$$

A sequence $\{x_n\}$ is called ***bounded*** if there exists a number $M > 0$ such that for any $n \in N$ the inequality $|x_n| \le M$ holds.

Otherwise the sequence is called ***unbounded***. It is easy to see that the sequences y_n and u_n are bounded, and v_n and z_n are unbounded.

A sequence is called ***increasing (non-decreasing)*** if the next inequality holds for any $n \in N$: $x_{n+1} > x_n \ (x_{n+1} \ge x_n)$.

Similary, a ***decreasing (non-increasing)*** sequence is defined. All these sequences are called

monotone sequences. The sequences v_n, y_n and u_n are monotonic, and z_n is the non-monotonic sequence.

If all elements of the sequence are equal to the same number c, then it is called a constant sequence. Another way to set a sequence is the ***recurrent*** way to set a sequence. It sets the initial element (the first member of the sequence) and the rule for determining n-th element by (n-1)-th.

$x_n = f\left(x_{n-1}\right).$

Thus, $x_2 = f\left(x_1\right), x_3 = f(x_2),...$

With this method of setting the sequence to determine the 100-th element, you must first count the first 99-th elements.

7.2 Limit of a numerical sequence.

You can see that the members of the sequence u_n approach to the number 1 indefinitely. In this case, it is said that the sequence of u_n, $n \in N$ tends to the limit of 1.

A number ***a*** is called the ***limit of the sequence*** $\{x_n\}$ if, for any arbitrarily small positive number ε, there is such a finite number N, depending on ε, that the inequality $\left|x_n - a\right| < \varepsilon$ holds for all n> N .

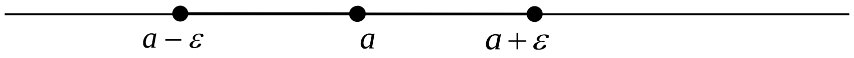
In this case, write $\lim\limits_{n\to\infty} x_n = a$ or $x_n \underset{n\to\infty}{\to} a$ and say that the sequence $\{x_n\}$ has a limit equal to the number *a*.

It is also said that the sequence of $\{x_n\}$ converges to *a*.

In short the definition of the limit can be written as follows:

$(\forall \varepsilon > 0\ \exists N(\varepsilon) : \forall n > N \Rightarrow \left|x_n - a\right| < \varepsilon) \Leftrightarrow \lim\limits_{n\to\infty} x_n = a\,.$

Let's find out the geometric meaning of determining the limit for the sequence. Inequality $\left|x_n - a\right| < \varepsilon$ is equivalent to the inequalities $-\varepsilon < x_n - a < \varepsilon$ or $a - \varepsilon < x_n < a + \varepsilon$ that shows that the element x_n is in the ε- vicinity of the point *a*.



Therefore, the definition of the limit of the sequence can be formulated geometrically as follows: the number a is called the limit of a sequence $\{x_n\}$, if for any ε - vicinity of the point *a*, there is a natural number N, such that all values of x_n for wich n> N fall into the ε - vicinity(neighborhood) of the point *a*.

It is clear that the smaller ε, the greater N, but in any case, there are infinite number of members of the sequence inside the ε - neighborhood of the point *a*, and only *a* finite number of them can be outside of it.

It follows that a convergent secuence has only one limit.

A secuence that has no limit is called a divergent secuence.

Such is, for example, the secuence $\{v_n\}$.

The constant secuence $\{x_n\} = \{C\}, n \in N$ has a limit equal to C.

Indeed, for $\forall \varepsilon > 0$ for all natural n, the inequality $|x_n - C| = |C - C| = 0 < \varepsilon$ holds.

And this, according to the definition of the limit, means that $\lim\limits_{n \to \infty} \{x_n\} = \lim\limits_{n \to \infty} C = C$.

7.3 Properties of the limit in inequalities.

Consider the secuences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$.

Theorem 7.1. If $\lim\limits_{n \to \infty} x_n = a$, $\lim\limits_{n \to \infty} y_n = b$ and, starting from some number, the inequality $x_n \le y_n$ is satisfied, then $a \le b$.($\lim\limits_{n \to \infty} x_n \le \lim\limits_{n \to \infty} y_n$) .

Proof.

Let's assume that $a > b$. From the equality $\lim\limits_{n \to \infty} x_n = a$ and $\lim\limits_{n \to \infty} y_n = b$ it follows, that for any $\varepsilon > 0$, there is such a natural number $N(\varepsilon)$ that for all $n > N$ the inequality $|x_n - a| < \varepsilon$ and $|y_n - b| < \varepsilon \Leftrightarrow a - \varepsilon < x_n < a + \varepsilon$, $b - \varepsilon < y_n < b + \varepsilon$ will be satisfied.

Take $\varepsilon = \frac{a-b}{2}$. Than:

$$x_n > a - \varepsilon = a - \frac{a-b}{2} = \frac{a+b}{2} \Leftrightarrow x_n > \frac{a+b}{2}$$ and

$$y_n < b + \varepsilon = b + \frac{a-b}{2} = \frac{a+b}{2} \Leftrightarrow y_n < \frac{a+b}{2}.$$

Therefore, $x_n > y_n$. This contradicts the condition, that $x_n \le y_n$. Therefore, $a \le b$.

Theorem 7.2. If $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = a$ and the inequality $x_n \le z_n \le y_n$ is valid starting from some number N, then $\lim\limits_{n\to\infty} z_n = a$.

7.4 The limit of a monotone bounded sequence. Number e. Natural logarithms.

Not every sequence has a limit. We formulate without proof the condition of the existence of the limit for the sequence.

Weierstras´s Theorem 7.3. Any monotone and bounded sequence has a limit.

As an example of the application of this feature, consider the sequence $x_n = \left(1+\frac{1}{n}\right)^n$, $n \in N$.

By the Newton binomial formula

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-(n-1))}{n!}b^n$$

Assuming $a = 1, b = \frac{1}{n}$, we get

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}\ldots + \frac{n(n-1)(n-2)\ldots(n-(n-1))}{n!}\cdot\frac{1}{n^n} =$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right)$$ ore

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right) \quad (7.3)$$

$$1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right)\le 2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}<$$

$$<1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}}\right)=1+\frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}=1+2\left(1-\frac{1}{2^n}\right)<1+2\cdot 1=3$$

$\Rightarrow 2<\left(1+\frac{1}{n}\right)^n<3$. That is, the sequence is bounded. We show that it is monotonically increasing. It follows from the equation (7.3) that as *n* increases, the number of positive terms in the right-hand side increases. Therefore, the sequence is increasing and $\left(1+\frac{1}{n}\right)^n>2$.

So, the sequence is bounded and the inequality holds for all *n*

$$2<\left(1+\frac{1}{n}\right)^n<3.$$

Therefore, based on the Weierstras´s theorem, the sequence $x_n=\left(1+\frac{1}{n}\right)^n$, $n\in N$ has a limit, usually denoted by the letter *e*.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e \qquad (7.6)$$

The number *e* is called Neper's number. The number *e* is irrational its approximate value is 2,72 (e=2,71828…).

§8. Function limit.

8.1. Limit of the function at a point.

Let the function *f(x)* be defined in some neighborhood of the point x_0, exept, perhaps, the point x_0 itself.

We formulate two equivalent among themselves definitions for the limit of a function at a point.

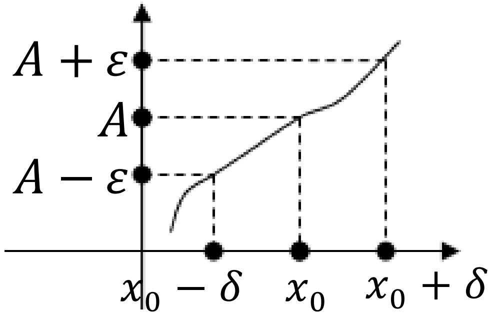
Definition 1. (in the language of sequence, or by Heine). The number A is called the limit of the function *f(x)* at the point x_0 (or at $x \to x_0$), if for any sequence of valid values of the argument $x_n,\ n \in N\ (x_n \neq x_0)$ converginging to x_0, the sequence of corresponding values of the function $f(x_n),\ n \in N$, converges to the number A. In this case, write $\lim\limits_{x \to x_0} f(x) = A$ or $f(x) \to A$ at $x \to x_0$.

Geometric meaning of the function limit: $\lim\limits_{x \to x_0} f(x) = A$ means that for all points *x* that are sufficiently close to the point x_0, the corresponding values of the function $f(x)$ are in the small neighborhood of the number A.

Definition 2. (in ε - δ language or by Cauchy). The number A is called the limit of the function at the point x_0 (or at $x \to x_0$), if for all arbitrarily small positive number ε there is such a positive number δ, depending on ε, that for all *x* satisfying the inequality $|x - x_0| < \delta$, the inequality $|f(x) - A| < \varepsilon$ is satisfied. Write down $\lim\limits_{x \to x_0} f(x) = A$.(In short it can be written as follows).

This definition can be briefly written as $(\forall \varepsilon > 0) \exists \delta(\varepsilon) > 0,\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - A| < \varepsilon.$

The geometric meaning of the limit for the function $A = \lim\limits_{x \to x_0} f(x)$: if for any ε > 0 neighborhood of point A there is δ(ε) > 0 neighborhood of point x_0, that for all $x = x_0$ from this δ – neighborhood, the corresponding values of the function $f(x)$ lies in the interval $(A - \varepsilon;\ A + \varepsilon)$



8.2. One-sided limits.

In determining the limit of the function *f(x)*, it is assumed that *x* tends to x_0 in any way: remaining smaller that x_0 (to the left of x_0), larger than x_0 (to the right of x_0), or fluctuating near the point x_0.

There are cases when the method of approximation of the argument *x* to x_0 significantly affects on the value of the limit of the function. Therefore, the concept of one-sided limits is introduced.

The number A_1 is called the limit of the function on the left at the point x_0, if for any arbitrarily small positive number ε > 0 there is such a positive number δ(ε) depending on ε that as soon as $x \in (x_0-\delta,\ x_0)$ the next non-equality holds: $|f(x)-A_1|<\varepsilon.$

Briefly, $\lim\limits_{x\to x_0-0} f(x)=A_1 \Leftrightarrow \forall\varepsilon>0\ \exists\delta(\varepsilon): \forall x\in(x_0-\delta,\ x_0) \Rightarrow |f(x)-A_1|<\varepsilon.$

Similary, the limit of the function on the right is defined:

$\forall\varepsilon>0\ \exists\delta(\varepsilon): \forall x\in(x_0,\ x_0+\delta) \Rightarrow |f(x)-A_2|<\varepsilon \Leftrightarrow \lim\limits_{x\to x_0+0} f(x)=A_2.$

The limits of the function on the left and the right are called one-sided limits.

Note, that if there is $\lim\limits_{x\to x_0} f(x)=A$, then there are also both $\lim\limits_{x\to x_0-0} f(x)=A_1$ and $\lim\limits_{x\to x_0+0} f(x)=A_2$. Moreover $A=A_1=A_2$.

The opposite statement is also true: if there are both one-sided limits $\lim\limits_{x\to x_0-0} f(x)=A$, $\lim\limits_{x\to x_0+0} f(x)=A$ and they are equal, then there is a limit at the point x_0 $\lim\limits_{x \to x_0} f(x) = A$ equal to the value of the one-sided limits of the function $f(x)$.

If $A_1 \neq A_2$, then the limit of the function at the point x_0 does not exist.

8.3. The limit of the function at *x* tending to infinity ($x \to \infty$).

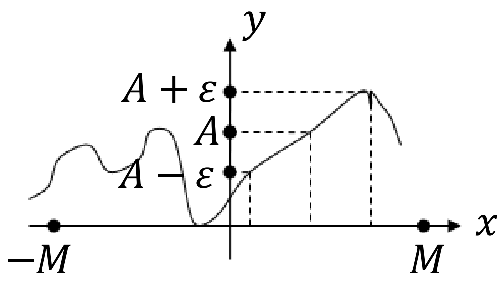
Let the function $y = f(x)$ be defined in the interval $(-\infty, +\infty)$.

A number A is called the limit of the function for $x \to \infty$, if for any arbitrarily positive number ε, there exists a number $M(\varepsilon) > 0$ that depends on ε, such that as soon as $|x| > M$ the next inequality holds: $|f(x) - A| < \varepsilon$.

In short, this definition can be written as : $\forall \varepsilon > 0$ $\exists$ M(ε) > 0: $\forall x$ $|x| > M \Rightarrow$ $|f(x) - A| < \varepsilon \Leftrightarrow$

$\lim\limits_{x \to +\infty} f(x) = A$, $\lim\limits_{x \to -\infty} f(x) = A$.

Geometrically, the meaning of this definition is as follows: for $\forall \varepsilon > 0$ $\exists$ M(ε) > 0 that at $|x| > M \Leftrightarrow$ $x \in (-\infty, -M) \cup (M, +\infty)$ the corresponding values of the function $f(x)$ fall into ε-neighborhood of the point A.



8.4. Infinitely large functions (i l f).

A function $y = f(x)$ is called infinitely large for $x \to x_0$, if for any number $M > 0$ there is a number $\delta(M) > 0$ such that for all x sitisfying the inequality $0 < |x - x_0| < \delta$, the inequality $|f(x)| > M$ is satisfied. This can be written as $\lim\limits_{x \to x_0} f(x) = \infty$ or $f(x) \to \infty$ at $x \to x_0$.

In short it can be written as follows:

$\forall\ M > 0\ \exists\delta(M) > 0\ \forall x : 0 < |x - x_0| < \delta \to |f(x)| > M \Leftrightarrow \lim\limits_{x\to x_0} f(x) = \infty$.

For example, the function $\dfrac{1}{x+5}$ is an infinity large at $x \to -5$.

If the function tends to infinity at $x \to x_0$ and takes positive values in the δ – neighborhood of the point x_0, then we write $\lim\limits_{x\to x_0} f(x) = +\infty$.

If the function tends to infinity at $x \to x_0$ and takes negative values in the δ – neighborhood of the point x_0, then we write $\lim\limits_{x\to x_0} f(x) = -\infty$.

The function $y = f(x)$, given on the entire number line, is called infinitely large for $x \to \infty$, if for any $M > 0$ there is such a number $N(M) > 0$ that for all *x* satisfying the inequality $|x| > N$, the inequality $|f(x)| > M$ is satisfied.

For example $y = \operatorname{tg} x$ is an infinitely large function with *x* tending to $\dfrac{\pi}{2}$, $\left(\lim\limits_{x\to\frac{\pi}{2}} \operatorname{tg} x = \infty\right)$.

Note, that if the argument *x*, tending to the infinity, take only natural values, i.e. $x \in N$, then the corresponding infinitely large function becomes an infinitely large sequence.

It is easy to see that every infinitely large function in the neighborhood of the point x_0 is unbounded in this neighborhood. The converse is not true: an unbounded function may not be an infinitely large function. (For example $y = x\sin x$).

However, if $\lim\limits_{x\to x_0} f(x) = A$, where A is a finite number, then the function $y = f(x)$ is bounded in some neighborhood of the point x_0.

Indeed, from the definition of the limit of the function, it follows that for $x \to x_0$ the condition $|f(x) - A| < \varepsilon$ is satisfied.

$\Rightarrow A - \varepsilon \le f(x) \le A + \varepsilon\ \ \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

This means that the function $f(x)$ is bounded.

§9. Infinitely small functions

9.1. Definitions and basic theorems.

The function $f(x) = y$ is called infinitely small for $x \to x_0$, if $\lim\limits_{x \to x_0} f(x) = 0$. (9.1)

According to the limit definition, this means that $\forall \varepsilon > 0\ \exists \delta(\varepsilon) : \forall x,\ 0 < |x - x_0| < \delta \Rightarrow |f(x)| < \varepsilon$.

Similarly, an infinitely small function is defined for $x \to x_0 + 0$, $x \to x_0 - 0$, $x \to +\infty$, $x \to -\infty$. In all these cases $f(x) \to 0$.

Examples of infinitely small functions:

$y = x^2$ for $x \to 0$, $y = x - 2$ for $x \to 2$, $y = \sin x$ for $x \to \pi k, k \in Z$.

Theorem 9.1. The algebraic sum of infinitely small functions is an infinitely small function.

Let $\alpha(x)$ and $\beta(x)$ be two infinite small functions $\left(\lim\limits_{x \to x_0} \alpha(x) = 0 \text{ and } \lim\limits_{x \to x_0} \beta(x) = 0 \right)'$.

According to the definition of the limit of the function

$$\forall \varepsilon > 0 \Rightarrow \forall \frac{\varepsilon}{2} > 0\ \exists\ \delta_1\left(\frac{\varepsilon}{2}\right) \text{ and } \delta_2\left(\frac{\varepsilon}{2}\right) \text{ such that } |\alpha(x)| < \frac{\varepsilon}{2},\ 0 < |x - x_0| < \delta_1\left(\frac{\varepsilon}{2}\right)$$

$$|\beta(x)| < \frac{\varepsilon}{2}\quad 0 < |x - x_0| < \delta_2\left(\frac{\varepsilon}{2}\right).$$

Let us define by $\delta = \min(\delta_1, \delta_2) \Rightarrow |\alpha(x)| < \frac{\varepsilon}{2}$ for all *x* satisfying $|x - x_0| < \delta$

$|\beta(x)| < \frac{\varepsilon}{2}$ for all *y* satisfying $|x - x_0| < \delta \Rightarrow |\alpha(x) + \beta(x)| \le |\alpha(x)| + |\beta(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

Thus, $\forall \varepsilon > 0\ \exists\ \delta : \forall x,\ 0 < |x - x_0| < \delta \Rightarrow$

$|\alpha(x)+\beta(x)|<\varepsilon \Leftrightarrow \lim\limits_{x\to x_0}(\alpha(x)+\beta(x))=0$.

That is $\alpha(x)+\beta(x)$ - is an infinite small function.

Theorem 9.2. The product of an infinitely small function by a bounded function is an infinitely small function. (If $\lim\limits_{x\to x_0}\alpha(x)=0$. $g(x)\le M\ \forall x\in R\Rightarrow$ $\lim\limits_{x\to x_0}\alpha(x)\cdot g(x)=0$).

Inverstigation 9.1. Since every infinitely small function is bounded, then the product of two infinitely small functions is an infinitely small function.

(If $\lim\limits_{x\to x_0}\alpha(x)=0$, $\lim\limits_{x\to x_0}\beta(x)=0\Rightarrow \lim\limits_{x\to x_0}(\alpha(x)\cdot\beta(x))=0$).

Inverstigation 9.2. The product of an infinitely small function by a number is an infinitely small function.

(If $\lim\limits_{x\to x_0}\alpha(x)=0$. $\gamma=const\Rightarrow \lim\limits_{x\to x_0}\gamma\cdot\alpha(x)=0$).

Theorem 9.3. The quotient of the division of an infinitely small function at the point x_0 by a function that has a non-zero limit at the same point is an infinitely small function

($\lim\limits_{x\to x_0}\alpha(x)=0$, $\lim\limits_{x\to x_0}f(x)\ne 0\Rightarrow \lim\limits_{x\to x_0}\dfrac{\alpha(x)}{f(x)}=0$).

Lets $\lim\limits_{x\to x_0}\alpha(x)=0$ and $\lim\limits_{x\to x_0}f(x)=a\ne 0$. The function $\dfrac{\alpha(x)}{f(x)}=\dfrac{1}{f(x)}\cdot\alpha(x)$, where $\alpha(x)$ is infinitly small function and $\dfrac{1}{f(x)}$ is a bounded function. Then according to the theorem 9.2 the product of these functions is the infinitely small function.

Theorem 9.4. If $\alpha(x)$ is infinitely small function at the point x_0 ($\alpha(x) \neq 0$), then the inverse function $\frac{1}{\alpha(x)}$ is an infinitely large function at the same point x_0.

And per turnover: if the function is infinitely large at the point x_0, then its inverse function is infinitely small at this point.

Let $\lim_{x \to x_0} \alpha(x) = 0$ and $\lim_{x \to x_0} \beta(x) = 0$. Then

$\forall \varepsilon > 0 \, \exists \, \delta : \forall x \;\; 0 < |x - x_0| < \delta \Rightarrow \; |\alpha(x)| < \varepsilon$.

Then $\left|\frac{1}{\alpha(x)}\right| > \frac{1}{\varepsilon} \Leftrightarrow \left|\frac{1}{\alpha(x)}\right| > M$ where $M = \frac{1}{\varepsilon}$.

Therefore, $\alpha(x)$ is an infinitely large function.

Similarly, the converse statement could be proved.

Remark. The proofs of the theorems were carried out for the case $x \to x_0$, but they are also valid for the case $x \to \infty$.

9.2. Relationships between a function, its limit and an infinity small function.

Theorem 9.5. If a function has a limit equal to A, then it can be represented as the sum of a number A and an infinitely small function.

Proof. That is, if $\lim_{x \to x_0} f(x) = A$, than $f(x) = A + \alpha(x)$, where $\lim_{x \to x_0} \alpha(x) = 0$.

Let $\lim\limits_{x\to x_0} f(x)=A$, $\Rightarrow \forall\varepsilon>0\ \exists\ \delta(\varepsilon):\forall x\ \ 0<|x-x_0|<\delta\Rightarrow |f(x)-A|<\varepsilon\Leftrightarrow$

$|(f(x)-A)-0|<\varepsilon\Leftrightarrow \lim\limits_{x\to x_0}(f(x)-A)=0=\lim\limits_{x\to x_0}\alpha(x)$

$\Rightarrow f(x)-A=\alpha(x)\Rightarrow f(x)=A+\alpha(x)$. The proof is finished.

Theorem 9.6. (inverse theorem). If the function $f(x)$ can be represented as the sum of the number A and the infinitely small function $\alpha(x)$, then the number A is the limit of the function $f(x)$ for $x\to x_0$.

That is, if $f(x)=A+\alpha(x)$, then $\lim\limits_{x\to x_0} f(x)=A$. Indeed, let $f(x)=A+\alpha(x)$, where $\lim\limits_{x\to x_0}\alpha(x)=0$.

Then $\forall\varepsilon>0\ \exists\ \delta:\forall x,\ 0<|x-x_0|<\varepsilon\ |\alpha(x)|<\varepsilon\Leftrightarrow |f(x)-A|<\varepsilon$

$\Rightarrow \lim\limits_{x\to x_0} f(x)=A$.

9.3. Basic limit theorems.

Consider theorems that make it easier to find the limit of a functions. Formulations and proofs of these theorems for the cases when $x\to x_0$ and $x\to\infty$, are similar.

Theorem 9.7. The limit of the sum(subtraction) of two functions is equal to the sum(subtraction) of the limits of these functions.

Indeed, let $\lim\limits_{x\to x_0} f(x)=A$, $\lim\limits_{x\to x_0}\varphi(x)=B$. Then, according to the Theorem 9.5 on the relation of a function, its limit, and the infinitely small function, we can write:

$f(x)=A+\alpha(x)$, $\varphi(x)=B+\beta(x)$, where $\lim\limits_{x\to x_0}\alpha(x)=0$, $\lim\limits_{x\to x_0}\beta(x)=0$.

Then, $f(x)+\varphi(x)=A+B+(\alpha(x)+\beta(x))$ where

$\lim\limits_{x\to x_0}(\alpha(x)+\beta(x))=\lim\limits_{x\to x_0}\alpha(x)+\lim\limits_{x\to x_0}\beta(x)=0\Rightarrow$ (the theorem 9.6) $\lim\limits_{x\to x_0}(f(x)+\varphi(x))=A+B=\lim\limits_{x\to x_0}f(x)+\lim\limits_{x\to x_0}\varphi(x)$.

In the case of the subtraction of the functions, the proof is analogous. The theorem holds for the algebraic sum of any finite number of functions.

Corollary. A function can have only one limit when $x\to x_0$.

Proof. Indeed, let $\lim\limits_{x\to x_0}f(x)=A$, $\lim\limits_{x\to x_0}f(x)=B$. Than, according to the theorem 9.7 we have

$$0=\lim_{x\to x_0}(f(x)-f(x))=\lim_{x\to x_0}f(x)-\lim_{x\to x_0}f(x)=A-B\Rightarrow A-B=0\Rightarrow A=B.$$

The proof is finished.

Theorem 9.8. The limit of the product of two functions is equal to the product of the limits of these functions.

Proof. Indeed, let $\lim\limits_{x\to x_0}f(x)=A$, $\lim\limits_{x\to x_0}\varphi(x)=B$. Then $f(x)=A+\alpha(x)$, $\varphi(x)=B+\beta(x)$, where $\lim\limits_{x\to x_0}\alpha(x)=0$, $\lim\limits_{x\to x_0}\beta(x)=0$

$$\Rightarrow f(x)\cdot\varphi(x)=(A+\alpha(x))(B+\beta(x))=$$

$$=AB+(A\beta(x)+B\alpha(x)+\alpha(x)\beta(x))=AB+0=AB=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}\varphi(x).$$

$(\lim\limits_{x\to x_0}(A\beta(x)+B\alpha(x)+\alpha(x)\beta(x))=0)$.

The proof is finished.

Note, that the theorem is valid for the product of any finite number of functions.

Corollary. The constant multiplier can be taken out of the limit sign.

Indeed, $\lim\limits_{x\to x_0}(C\cdot f(x))\Rightarrow$ (the theorem 9.8)

$$\lim_{x\to x_0}(C\cdot f(x))=\lim_{x\to x_0}C\cdot\lim_{x\to x_0}f(x)=C\cdot\lim_{x\to x_0}f(x).$$

Similarly, the limit of degree whith a natural exponent is equal to the same degree of the limit.

Indeed, $\lim\limits_{x\to x_0}\left(f(x)\right)^n=\lim\limits_{x\to x_0}f(x)\cdot\lim\limits_{x\to x_0}f(x)\cdot\ldots\lim\limits_{x\to x_0}f(x)=\left(\lim\limits_{x\to x_0}f(x)\right)^n$.

Theorem 9.9. The limit of the fraction is equal to the ratio of the limit of the numerator to the limit of the denominator, provided that the limit of the denominator is different from zero at this point.

Indeed, let $\lim\limits_{x\to x_0} f(x)=A,\ \lim\limits_{x\to x_0}\varphi(x)=B\neq 0 \Rightarrow\ f(x)=A+\alpha(x),\ \varphi(x)=B+\beta(x)$

$$\Rightarrow \frac{f(x)}{\varphi(x)}=\frac{A+\alpha(x)}{B+\beta(x)}=\frac{A}{B}+\left(\frac{A+\alpha(x)}{B+\beta(x)}-\frac{A}{B}\right)=\frac{A}{B}+\frac{B\cdot\alpha(x)-A\cdot\beta(x)}{B^2+B^2\beta(x)}=\frac{A}{B}+\gamma(x)$$

$\lim\limits_{x\to x_0}\dfrac{B\cdot\alpha(x)-A\cdot\beta(x)}{B^2+B^2\beta(x)}=0=\gamma(x)$, where $\lim\limits_{x\to x_0}\gamma(x)=0$

$$\Rightarrow \lim_{x\to x_0}\frac{f(x)}{\varphi(x)}=\frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0}\varphi(x)}=\frac{A}{B}.$$

Example. Calculate.

9.3. $\lim\limits_{x\to 1}(3x^2-2x+7)=\lim\limits_{x\to 1}3x^2-\lim\limits_{x\to 1}2x+\lim\limits_{x\to 1}7=3\lim\limits_{x\to 1}3x^2-2\lim\limits_{x\to 1}x+7=3\cdot 1^2-2\cdot 1+7=3-2+7=8.$

9.4. $\lim\limits_{x\to 2}\dfrac{x^2+14x-32}{x^2-6x+8}=\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 2}\dfrac{(x-2)(x+16)}{(x-2)(x-4)}=\lim\limits_{x\to 2}\dfrac{x+16}{x-4}=\dfrac{18}{-2}=-9$

If the numerator and denominator are substituted with the value $x=2$, then we get an ucertaintly of the form zero divided by zero.

In order to get rid of this unertainty, the numerator and denominator are decomposed into the simplest factors. Reducing the same factors allows us to get rid of the uncertainty and get the final result

9.5. $\lim\limits_{x\to\infty}\dfrac{2x^2+3x+1}{4x^2+2x+5}=\left(\dfrac{\infty}{\infty}\right)=\lim\limits_{x\to\infty}\dfrac{2+\frac{3}{x}+\frac{1}{x^2}}{4+\frac{2}{x}+\frac{5}{x^2}}=\dfrac{2}{4}=\dfrac{1}{2}.$

$\lim\limits_{x\to\infty}\dfrac{3}{x}=0,\ \ \lim\limits_{x\to\infty}\dfrac{1}{x^2}=0,\ \ \lim\limits_{x\to\infty}\dfrac{2}{x}=0,\ \ \lim\limits_{x\to\infty}\dfrac{5}{x^2}=0.$

9.4. Signs of the existence of a limit.

Not every function, even a limited one, has a limit. For example, the function $y = \sin x$ at $x \to \infty$ has no limit. In many questions of analysis, it is sufficient only to make sure that there is a limit.

In such cases, the signs of the existence of a limit are used.

Theorem 9.10. (about the limit of the intermediate function).

If a function $f(x)$ is enclosed between two functions $\varphi(x)$ and $g(x)$ tend to the same limit, then it is also tend to this limit.

That is, if $\lim\limits_{x \to x_0} \varphi(x) = A, \lim\limits_{x \to x_0} g(x) = A$ and (9.6)

$\varphi(x) \le f(x) \le g(x)$, then $\lim\limits_{x \to x_0} f(x) = A$. (9.7)

Indeed, from equality (9.6) it follows that, for any $\varepsilon > 0$ there are δ_1 and δ_2 for the point x_0, in one of wich the inequality holds

$$|\varphi(x) - A| < \varepsilon \Leftrightarrow -\varepsilon < \varphi(x) - A < \varepsilon \qquad (9.8)$$

$$|g(x) - A| < \varepsilon \Leftrightarrow -\varepsilon < g(x) - A < \varepsilon \qquad (9.9)$$

Let $\delta = \min(\delta_1, \delta_2)$, then in δ - neighborhoot of the point x_0, both inegulities (9.8) and (9.9) are satisfied.

We have (9.7) $\varphi(x) \le f(x) \le g(x)$ then $\varphi(x) - A \le f(x) - A \le g(x) - A$ which is equivalent to

$$|f(x) - A| \le |g(x) - A| < \varepsilon \qquad (9.10)$$

$$\Rightarrow \forall \varepsilon > 0 \ \exists \ \delta(\varepsilon) \ \forall x \ 0 < |x - x_0| < \delta \ \ |f(x) - A| < \varepsilon \Rightarrow$$

$$\lim_{x \to x_0} f(x) = A.$$

Theorem 9.11. (on the existence of the limit for a monotone function).

If a function $f(x)$ is monotonic and bounded at $x < x_0$ or at $x > x_0$, then there is a left limit $\lim\limits_{x \to x_0 - 0} f(x) = f(x_0 - 0)$ or right limit $\lim\limits_{x \to x_0 + 0} f(x) = f(x_0 + 0)$ respectively.

Corollary. A bounded monotone sequence $\{x_n\},\ n \in N$ has a limit.

9.5. The first remarkable limit.

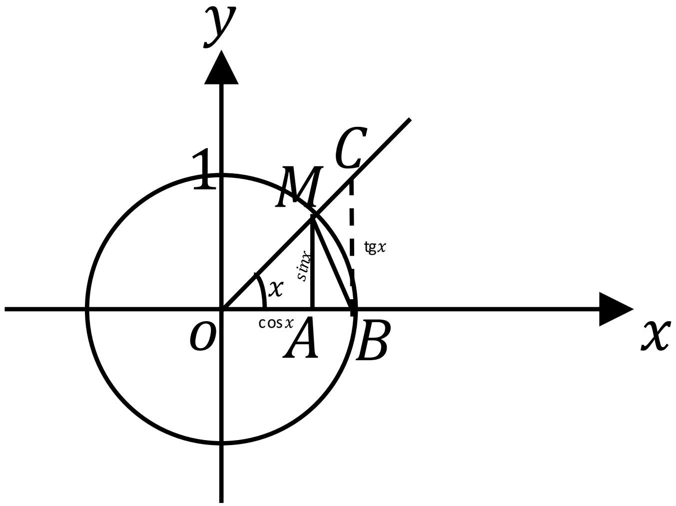
$$\lim_{x \to 0} \frac{\sin x}{x} = 1 .$$

In other words, the limit of the ratio of the sin of the variable to its variable, provided that the variable tends to zero, equal to one.

Indeed, take a circle of radius 1 and denote the radianum of the angle MOB by *x*. Let $0 < x < \frac{\pi}{2}$. In the drawing $|AM| = \sin x$, the arc MB is numerically equal to the central angle *x*, $|BC| = \operatorname{tg} x$. It is easy to see that $S_{\Delta MOB} < S_{\text{sector } MOB} < S_{\Delta COB}$.

Based on the corresponding trigonometric formulas we get

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\operatorname{tg} x \Big| : \text{ - divide by } \left(\frac{1}{2}\sin x > 0\right)$$



We get $1<\frac{x}{\sin x}<\frac{1}{\cos x}\Leftrightarrow 1<\frac{\sin x}{x}<\cos x$. Since $\lim\limits_{x\to 0}\cos x=1$ and $\lim\limits_{x\to 0}1=1$, then by the sign about the limit of intermediate function $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$.

Example. Find the next limits:

9.6. $\lim\limits_{x\to 0}\frac{\sin 3x}{2x}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{(\sin 3x)\cdot 3x}{3x\cdot 2x}=\frac{3}{2}$.

9.7. $\lim\limits_{x\to 0}\frac{\operatorname{tg} x}{x}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{\sin x}{x\cdot\cos x}=\frac{\lim\limits_{x\to 0}\frac{\sin x}{x}}{\lim\limits_{x\to 0}\cos x}=\frac{1}{1}=1$.

As a consequence of the first remarkable limit: $\lim\limits_{x\to 0}\frac{\operatorname{arctg} x}{x}=1$. $\lim\limits_{x\to 0}\frac{\arcsin x}{x}=1$.

9.6. The second remarkable limit.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e. \qquad (9.15)$$

Indeed: $\forall x\in R: n<x<n+1, n\in N$, where $n=[x]$ - the whole part of *x*.

Then $\frac{1}{n+1}\le\frac{1}{x}\le\frac{1}{n}$, $1+\frac{1}{n+1}\le 1+\frac{1}{x}\le 1+\frac{1}{n}$, $\left(1+\frac{1}{n+1}\right)^{n}\le\left(1+\frac{1}{x}\right)^{x}\le\left(1+\frac{1}{n}\right)^{n+1}$

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)}\le\left(1+\frac{1}{x}\right)^{x}\le\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right)$$

$$\lim_{x\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)}=\frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)}=\frac{e}{1}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=e\cdot 1=e\,.$$

Then, according to the theorem on the existence of the limit for the intermediate function

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e\,.$$

If we put $\frac{1}{x}=\alpha$ and at $x\to\infty$, $\frac{1}{\alpha}\to 0$, then (9.15) it will be written as

$$\lim_{\alpha\to 0}(1+\alpha)^{1/\alpha}=e\,. \qquad (9.16)$$

If the equality (9.16) is logarithmed by the base *e*, we get a logarithmic interpetation of the second remarkable limit.

$$\lim_{\alpha\to 0}\frac{\ln(1+\alpha)}{\alpha}=1 \qquad (9.17)$$

The equalities (9.15) – (9.17) are called the second remarkable limit.

Example. Find the next limit.

9.8. $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}=[1^{\infty}]=\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}\cdot\frac{2}{x}\cdot x}=e^{2}.$

§10. Equivalent infinitely small functions.

10.1. Comparison of infinitely small functions.

As you know, the sum, subtraction and the product of two infinity small functions is the infinitely small function. The ratio of two infinitely small functions can produce different results: be a finite number, be an infinitely large or infinitely small function, or do not tend to any limit at all.

Two infinitely small quantities are compared with each other by their ratio.

Let $\alpha(x)$ and $\beta(x)$ - are infinitely small function at the point x_0. That is: $\lim\limits_{x\to x_0} \alpha(x) = 0$, $\lim\limits_{x\to x_0} \beta(x) = 0$.

Definitions.

If $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)} = A \neq 0\ (A \in R)$, then $\alpha(x)$ and $\beta(x)$ are called infinitely small of the same order of smallness.

If $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)} = 0$, then $\alpha(x)$ is called infinitely small function(infinitely small) of a higher order of smallness than $\beta(x)$.

If $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)} = \infty$, then $\alpha(x)$ is called an infinitely small of a lower order of smallness than $\beta(x)$.

If $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)}$ does not exist, then $\alpha(x)$ and $\beta(x)$ are called incomparable infinitely small functions.

Note, that these are the same rules for comparing infinitely small functions for $x \to \pm\infty,\ x \to x_0 \pm 0$.

Examples. Compare infinitely small functions.

10.1. $\alpha(x) = 3x^2$, $\beta(x) = 14x^2$ at $x \to 0$. $\lim\limits_{x\to 0} \alpha(x) = 0$, $\lim\limits_{x\to 0} \beta(x) = 0$. It is easy to check that

$$\lim_{x\to 0} \frac{3x^2}{14x^2} = \frac{3}{14}.$$

Therefore, according to the definition 1, $\alpha(x)$ and $\beta(x)$ are infinitely small functions of the same order of smallness.

10.2. $\alpha(x) = 3x^4$, $\beta(x) = 7x$ at $x \to 0$. $\lim\limits_{x\to 0}\alpha(x) = 0$, $\lim\limits_{x\to 0}\beta(x) = 0$. Then

$$\lim_{x\to 0}\frac{\alpha(x)}{\beta(x)} = \lim_{x\to 0}\frac{3x^4}{7x} = \lim_{x\to 0}\frac{3x^3}{7} = 0.$$

Therefore, according to the definition 2, $\alpha(x)$ is a infinitely small function of a higher order of smallness than $\beta(x)$.

10.3. $\alpha(x) = \mathrm{tg}\, x$, $\beta(x) = x^2$ at $x \to 0$. $\lim\limits_{x\to 0}\alpha(x) = 0$, $\lim\limits_{x\to 0}\beta(x) = 0$. Then

$$\lim_{x\to 0}\frac{\mathrm{tg}\, x}{x^2} = \lim_{x\to 0}\frac{\sin x}{x\cdot\cos x\cdot x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}\cdot\lim_{x\to 0}\frac{1}{x} = 1\cdot 1\cdot\infty = \infty.$$

Therefore, according to the definition 3, $\alpha(x)$ is a infinitely small function of a lower order of smallness than $\beta(x)$.

10.4. $\alpha(x) = x\sin\frac{1}{x}$ and $\beta(x) = x$ at $x \to 0$. $\lim\limits_{x\to 0} x\sin\frac{1}{x}$ as the product of infinitely small function by a bounded function. $\lim\limits_{x\to 0} x = 0$. But $\lim\limits_{x\to 0}\frac{\alpha(x)}{\beta(x)} = \lim\limits_{x\to 0}\frac{x\sin\frac{1}{x}}{x} = \lim\limits_{x\to 0}\sin\frac{1}{x}$. This limit does not exist due to the periodicity of the function sin*x*.

10.2. Equivalent infinitely small functions and basic theorems about them.

Among infinitely small function the equivalent infinitely small functions play a special role.

If $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)} = 1$, then this is denoted as $\alpha(x) \sim \beta(x)$. For example $\sin x \sim x$ for $x \to 0$ since $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$.The function $\mathrm{tg}\, x \sim x$, because $\lim\limits_{x\to 0}\frac{\mathrm{tg}\, x}{x} = 1$.

Theorem 10.1. The limit of the ratio of two infinitely small functions will not change if each of them or one of them is replaced by an equivalent infinitely small function.

Indeed, let $\alpha(x) \sim \alpha'$, $\beta(x) \sim \beta'$ for $x \to x_0$. Then

$$\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)} = \lim_{x\to x_0}\frac{\alpha(x)\cdot\alpha'\cdot\beta'}{\alpha'\cdot\beta(x)\cdot\beta'} = \lim_{x\to x_0}\frac{\alpha(x)}{\alpha'}\cdot\lim_{x\to x_0}\frac{\beta(x)}{\beta'}.$$

$\lim\limits_{x\to x_0}\frac{\alpha'}{\beta'}=1\cdot 1\cdot\lim\limits_{x\to x_0}\frac{\alpha'}{\beta'}$. That is $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=\lim\limits_{x\to x_0}\frac{\alpha'}{\beta'}$.

It is easy to check that $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta'}=\lim\limits_{x\to x_0}\frac{\alpha'}{\beta(x)}$.

Theorem 10.2. The subtraction of two infinitely small functions is the infinitely small function of a higher order of smallness then each of them.

Indeed, let $\alpha(x)\sim\beta(x)$ to $x\to x_0$.

Then $\lim\limits_{x\to x_0}\frac{\alpha(x)-\beta(x)}{\alpha(x)}=\lim\limits_{x\to x_0}\left(1-\frac{\beta(x)}{\alpha(x)}\right)=\lim\limits_{x\to x_0}1-\lim\limits_{x\to x_0}\frac{\beta(x)}{\alpha(x)}=1-1=0$.

Similarly, $\lim\limits_{x\to x_0}\frac{\alpha(x)-\beta(x)}{\beta(x)}=\lim\limits_{x\to x_0}\left(\frac{\alpha(x)}{\beta(x)}-1\right)=\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}-\lim\limits_{x\to x_0}1=1-1=0$.

Therefore, $\lim\limits_{x\to x_0}\frac{\alpha(x)-\beta(x)}{\beta(x)}=0$.

The converse is also true: if the difference between $\alpha(x)$ and $\beta(x)$ is an infinitely small function of a higher order of smallness than $\alpha(x)$ and $\beta(x)$, then $\alpha(x)$ and $\beta(x)$ are equivalent infinitely small functions. Indeed, since $\lim\limits_{x\to x_0}\frac{\alpha(x)-\beta(x)}{\alpha(x)}=0$, then $\lim\limits_{x\to x_0}\left(1-\frac{\beta(x)}{\alpha(x)}\right)=0$, that is $1-\lim\limits_{x\to x_0}\frac{\beta(x)}{\alpha(x)}=0\Rightarrow$ $\lim\limits_{x\to x_0}\frac{\beta(x)}{\alpha(x)}=1$. That is $\alpha(x)\sim\beta(x)$.

Similarly, if $\lim\limits_{x\to x_0}\frac{\alpha-\beta}{\beta}=1$ then $\alpha(x)\sim\beta(x)$.

Theorem 10. 3. The sum of a finite number of infinitely small functions of different orders is equivalent to the lowest-order term.

Proof. We prove the theorem for two functions.

Let $\alpha(x)\to 0$, $\beta(x)\to 0$ for $x\to x_0$, and $\alpha(x)$ is infinitely small function of higher order of smallness than $\beta(x)$, that is $\lim\limits_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=0$. Then

$$\lim_{x\to x_0}\frac{\alpha(x)+\beta(x)}{\beta(x)}=\lim_{x\to x_0}\left(\frac{\alpha(x)}{\beta(x)}+1\right)=\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)}+\lim_{x\to x_0}1=0+1=1.$$

Therefore, $\alpha(x)+\beta(x)\sim\beta(x)$ for $x\to x_0$.

The summand equivalent to the sum of infinitely small function is called the principal part of this sum.

Replacing the sum of infinitely small functions with its main part is called discarding infinitely small functions of higher order.

Example.

10.5. Find the limit: $\lim\limits_{x\to 0}\frac{3x+7x^2}{\sin 2x}=\lim\limits_{x\to 0}\frac{3x}{\sin 2x}=\lim\limits_{x\to 0}\frac{3x}{2x}=\frac{3}{2}$.

Since $3x+7x^2\sim 3x$ and $\sin 2x\sim 2x$ to $x\to 0$

10.3. Applying equivalent infinitely small functions to computing limits.

Calculating limits. To solve uncertainties of the form $\left(\frac{0}{0}\right)$, it is often useful to apply the principle of replacing infinitesimals with equivalent ones and other properties of infinitesimals as you know $\sin x\sim x$, $\operatorname{tg} x\sim x$ for $x\to 0$.

Here are some more examples of infinitely small functions.

Examples. 10.6. $\lim\limits_{x\to 0}\frac{1-\cos x}{\frac{x^2}{2}}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\frac{x^2}{4}}=\frac{2}{2}\lim\limits_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=1.$

Therefore, $1-\cos x\sim\frac{x^2}{2}$ at $x\to 0$.

10.7. $\lim\limits_{x\to 0}\frac{\arcsin x}{x}=\left(\frac{0}{0}\right)=\left|\begin{array}{c}\arcsin x=t\\ x=\sin(\arcsin x)=\sin t\end{array}\right|=\lim\limits_{t\to 0}\frac{t}{\sin t}=\lim\limits_{t\to 0}\frac{1}{\frac{\sin t}{t}}=\frac{1}{\lim\limits_{t\to 0}\frac{t}{\sin t}}=\frac{1}{1}=1.$

Therefore, $\arcsin x \sim x$.

10.8. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{\frac{x}{2}}=\left(\dfrac{0}{0}\right)=$

$$=\lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{\frac{x}{2}\left(\sqrt{1+x}+1\right)}=\lim_{x\to 0}\frac{2(1+x-1)}{x\left(\sqrt{1+x}+1\right)}=\lim_{x\to 0}\frac{2}{\sqrt{1+x}+1}=\frac{2}{2}=1.$$

Therefore, $\sqrt{1+x}-1 \sim \dfrac{x}{2}$.

The following are the most important equivalences that are used in calculating the limits.

1. $\sin x \sim x \ (x\to 0)$
2. $\operatorname{tg} x \sim x \ (x\to 0)$
3. $\arcsin x \sim x \ (x\to 0)$
4. $\operatorname{arctg} x \sim x \ (x\to 0)$
5. $1-\cos x \sim \dfrac{x^2}{2} \ (x\to 0)$
6. $e^x-1 \sim x \ (x\to 0)$
7. $a^x-1 \sim x\ln a \ (x\to 0)$
8. $\ln(1+x) \sim x \ (x\to 0)$
9. $\log_a(1+x) \sim x\log_a e \ (x\to 0)$
10. $(1+x)^k \sim 1+kx,\ k>0 \ (x\to 0)$
11. $\sqrt{1+x}-1 \sim \dfrac{x}{2} (x\to 0)$.

Examples. Find the next limits:

10.9. $\lim\limits_{x\to 0}\dfrac{\operatorname{tg}2x}{\sin 3x}=\lim\limits_{x\to 0}\dfrac{2x}{3x}=\dfrac{2}{3}$.

10.10. $\lim\limits_{x\to\infty}x\left(e^{\frac{1}{x}}-1\right)=\left|\dfrac{1}{x}=t,\ x=\dfrac{1}{t}\right|=\lim\limits_{t\to 0}\dfrac{e^t-1}{t}=\lim\limits_{t\to 0}\dfrac{t}{t}=1$.

10.11. $\lim\limits_{x\to 1}\dfrac{\arcsin(x-1)}{x^2-5x+4}=\lim\limits_{x\to 1}\dfrac{\arcsin(x-1)}{(x-1)(x-4)}=\lim\limits_{x\to 1}\dfrac{(x-1)}{(x-1)(x-4)}=\dfrac{1}{1-4}=-\dfrac{1}{3}$.

§11. Continuity of a function.

11.1. Continuity of a function at a point.

Let the function $y = f(x)$ be defined at the point x_0 and in some neigborhood of this point. A function $y = f(x)$ is called ***continuous*** at the point x_0 if there is a limit of the function at that point and it is equal to the value of the function at that point.

That is $\lim\limits_{x \to x_0} f(x) = f(x_0)$ (11.1)

Equality (11.1) means that three conditions are hold:

the function $f(x)$ is defined at the point x_0 and its neighborhood

the function $f(x)$ has a limit at the point x_0

the limit of the function at the point x_0 is equal to the value of the function at this point, that is, the equality (11.1) is satisfied.

Since $\lim\limits_{x \to x_0} x = x_0$, the equality (11.1) can be written as

$$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x) = f(x_0). \qquad (11.2)$$

This means that when finding the limit of a continuous function $y = f(x)$, you can go to the limit under the sign of the function, that is, in the function $y = f(x)$ instead of the argument x add its limit value x_0.

For example, $\lim\limits_{x \to 0} e^{\frac{\sin x}{x}} = e^{\lim\limits_{x \to 0} \frac{\sin x}{x}} = e^1 = e$.

In the first equality, the function and the limit are reversed(11.2) due to the continuity of the function e^x.

Example. Calculate.

11.1. $A = \lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x \to 0} \dfrac{1}{x} \cdot \ln(1+x) = \lim\limits_{x \to 0} \ln(1+x)^{\frac{1}{x}} = \ln\left(\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}}\right) = \ln e = 1$.

Or by virtue of the equivalence of infinitely small functions $\ln(1+x)$ and x at $x \to 0$:

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{x}{x} = 1.$$

We can give another definition of the continuity of a function at a point x_0, based on the concept of the increment of an argument and the function.

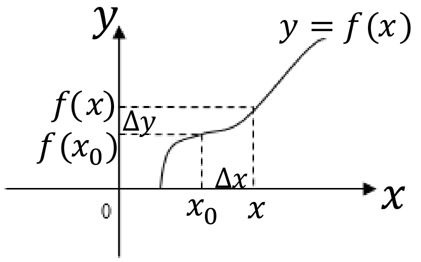
Let the function $y = f(x)$ be defined on some interval (a, b). Take an arbitrary point $x_0 \in (a;b)$. Fof any $x \in (a, b)$, the difference $(x - x_0)$ is called the increment of the argument x at the point x_0 and is denoted by $\Delta x : \Delta x = x - x_0$. Hence $x = x_0 + \Delta x$.

The difference between the corresponding values of the function $f(x) - f(x_0)$ is called the increment of the function $f(x)$ at the point x_0 and is denoted by $\Delta y : \Delta y = f(x) - f(x_0)$ or $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

Obviously, the increments Δx and Δy can be either positive or negative.

Let's write the equality (11.1) in the new notations. Since the conditions $x \to x_0$ and $x - x_0 \to 0$ are the same, then the equation (11.1) takes the form

$$\lim_{x\to x_0}(f(x) - f(x_0)) = 0 \text{ or } \lim_{\Delta x\to 0}\Delta y = 0. \quad (11.3)$$



The resulting equality is another definition of the continuity of a function $f(x)$ at the point x_0: a function $y = f(x)$ is called continuous at a point x_0 if it is defined at a point x_0 and its neighborhood and equality (11.3) holds, that is an infinitesimal increment of the argument corresponds to an infinitesimal increment of the function.

When investigating the continuity of a function at a point use either the first (11.1) or second (11.3) definition.

Example.

Inverstigate the continuity of the function:

11.2. $y = \sin x$.

Solution. The function $y = \sin x$ is defined for all $x \in R$.

Take an arbitrary point *x* and find the increment Δy:

$\Delta y = \sin(x + \Delta x) - \sin x = 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$. Therefore,

$\lim\limits_{\Delta x \to 0} \Delta y = \lim\limits_{\Delta x \to 0} 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2} = 0$. Since the product of a bounded function $\cos x$ by an infinitesimal function $\sin\frac{\Delta x}{2}$ is an infinitesimal function.

According to the definition (11.3), the function $y = \sin x$ is continuous at the point *x*.

Similarly, we prove the continuity of the function $y = \cos x$.

11.3. Function classification.

The points of discontinuity of a function and their classification.

The points at which the continuity of the function is broken are called the break points of this function. The following classification of the break points has a place:

The limit of the function at a point exists, but the function *f(x)* it self is not defined at that point. For example, $y = \frac{\sin x}{x}$ is not defined at the point $x = 0$

, but the limit of the function at that point exists. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$. Such a break point of the function is called the point of the eliminated gap.

When the left and rigt limits of a function at the point x_0 exist, but are not equal to each other, then such a point x_0 is called a break point of the first kind.

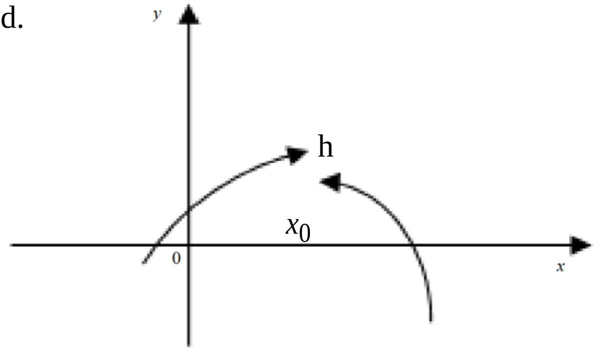


Fig. 11.1.

That is, there is a final jump in the change in the value of the function when passing through this point.

In the case when one of the left or right limits or both limits do not exist, there is an infinit jump of the function when passing through this point, then such a break point is called a break point of the second kind. (fig. 11.2).

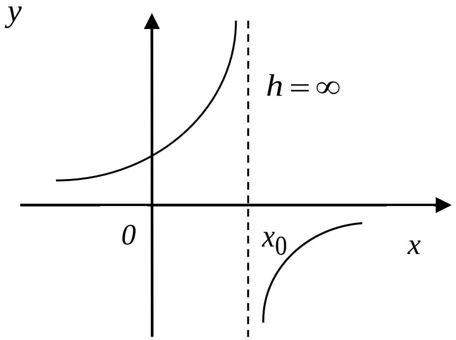


Fig. 11.2.

Example 11.1. Find the break points of the function and set their characters.

$f(x)=\frac{e^x-1}{x}$.

Solution. Since the function e^x-1 and x are continious at any point, their ratio will also be continuous at all points $x\neq 0$. For $x=0$ the function indeterminate. Hence, it is discontinuous. Since $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1, x=0$ is the point of the eliminated gap.

If we put $f(0)=1$, then the function

$\varphi(x)=\begin{cases} f(x), x\neq 0 \\ 1, x=0 \end{cases}$ will be continuous for all $x\in R$.

Example 11.2. Set the nature of the function break points

$$f(x)=\begin{cases} x^2+1, -\infty<x\leq 0 \\ x+1, 0<x\leq 3 \\ 6-x, x>3 \end{cases}$$

Solution. The functions definition area is the entire numeric axis $(-\infty;+\infty)$. Discontinuities of the function are possible only at the points $x=0$ and $x=3$, where the function changes its analytical task, its appearance. Find the oneway limits at the point $x=0$.

$\lim\limits_{x\to 0-0} f(x)=\lim\limits_{x\to 0-0}(x^2+1)=1$. $\lim\limits_{x\to 0+0} f(x)=\lim\limits_{x\to 0+0}(x+1)=1$, $f(0)=1$. We have $\lim\limits_{x\to 0-0} f(x)=\lim\limits_{x\to 0+0} f(x)=f(0)$.

Therefore, at the point $x=0$ the function is continuous.

$\lim\limits_{x\to 3-0} f(x)=\lim\limits_{x\to 3-0}(x+1)=4$. $\lim\limits_{x\to 3+0} f(x)=\lim\limits_{x\to 3+0}(6-x)=3$. $\Rightarrow$ at the point $x=3$ the function has a discontinuity of the first kind.

Example 11.3. Set the nature(character) of the function break points $f(x)=e^{\frac{1}{x+1}}$.

This function is continuous everywhere except for the point $x=-1$. Let us find the right and left limits for the function at this point:

$$\lim\limits_{x\to -1-0} f(x)=\lim\limits_{x\to -1-0} e^{\frac{1}{x+1}}=\lim\limits_{x\to -1-0} e^{\frac{1}{-1-0+1}}=\lim\limits_{x\to -1-0} e^{\frac{1}{-0}}=e^{-\infty}=\frac{1}{e^{\infty}}=0.$$

$$\lim_{x\to -1+0} f(x) = \lim_{x\to -1+0} e^{\frac{1}{x+1}} = \lim_{x\to -1+0} e^{\frac{1}{-1+0+1}} = \lim_{x\to -1+0} e^{\frac{1}{0}} = e^{\infty} = \infty .$$

At the point $x = -1$, we observe the infinity jump for the function. Hence $x = -1$ is the break point of the second kind (Fig. 11.2.1).

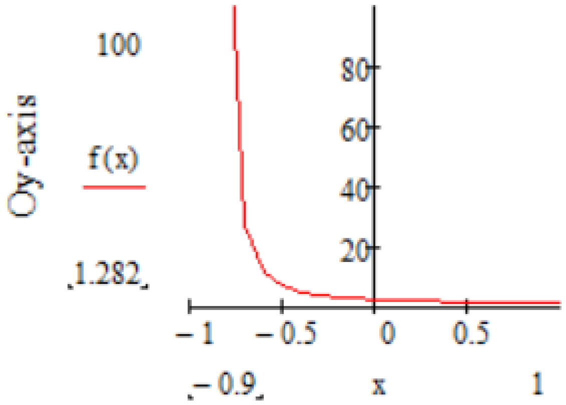


Fig. 11.2.1 The graph for the function $f(x) = e^{\frac{1}{x+1}}$.

11.4. Basic theorems about continuous functions. Continuity of elementary functions.

The continuity theorems of functions follow directly from the corresponding limit theorems.

Note thate a function is continuous on the closed segment $[a, b]$, if it is continuous at any point of this segment.

Theorem 11.1. The sum, product, quotient of two continuous function is a continuous function (for the quotient, except for those argument values in which the divisor is zero).

Theorem 11.2. Let the function $u = \varphi(x)$ be continuous at a point x_0, and let the function $y = f(u)$ be continuous at the point $u_0 = \varphi(x_0)$. Then, the complex function $y = f(\phi(x))$, consisting of continiuious functions, is the continuous function at the point x_0.

Proof. By virtue of the continuity of the function $u=\varphi(x)$, $\lim\limits_{x\to x_0}\varphi(x)=\varphi(x_0)=u_0$.

That is, at $x\to x_0$ we have $u\to u_0$.

Therefore due to the continuity of the function $y=f(u)$, we have

$$\lim_{x\to x_0} f(\varphi(x))=\lim_{u\to u_0} f(u)=f(u_0)=f(\varphi(x_0)).$$

This proves that the complex function $f(\varphi(x))$ is continuous at the point x_0.

Theorem 11. 3. If the function $y=f(x)$ is continuous and strictly monotone on the replaced segment $[a,b]$ of the axis Ox, then the inverse function $y=\varphi(x)$ is also continuous and monotone on the corresponding segment $[c,d]$ of the axis Oy.(The proof is umitted).

So, for example, the function $\operatorname{tg} x=\dfrac{\sin x}{\cos x}$, by virtue of theorem 11.1, is a continuous function for all values of x, except for those x for which $\cos x=0$

, that is, except for the value of $x=\dfrac{\pi}{2}+\pi n,\ n\in Z$.

It can be proved that all basic elementary functions are continuous for all values of x for which they are defined.

As we know, an elementary function is a function, that can be defined by single formula containing a finite number of arithmetic operations and operations for taking a function from a function (the operation of superposition for the functions) of the basic elementary functions.

Therefore, it follows from the above theorems: every elementary function is continuous at every point at which it is defined.

This important result makes it possible, in particular, to easily find the limit of elementary functions at the points where they are defined.

Example 11.4. Find the limit. Since the function is continuous at the point $x_0=\dfrac{\pi}{4}$

$$\lim_{x\to\frac{\pi}{4}} 2^{\operatorname{ctg} x}=2^{\lim\limits_{x\to\frac{\pi}{4}}\operatorname{ctg} x}=2^{\operatorname{ctg}\frac{\pi}{4}}=2^1=2.$$

11.5. Properties of functions that are continuous on a segment.

The function that is continuous on the segment have a number of important properties. Let us form them in the form of theorems without giving their proofs.

Theorem 11. 4. (**Weierstras´s Theorem).** If a function is continuous on a segment, then it reaches its largest and smallest values on this segment.

Corollary. If a function is continuous on a segment, then it is bounded on that segment.

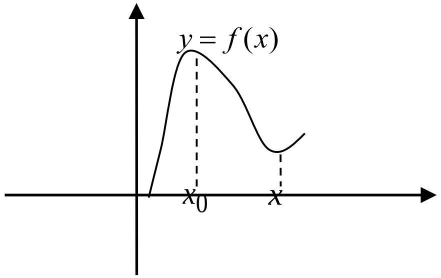


Fig. 11.3

Theorem 11. 5. (**The Bolzano-Cauchy Theorem).** If a function is continuous on a segment and takes at its ends unequal values of A and B, then on this segment it takes all intermediate values between A and B.

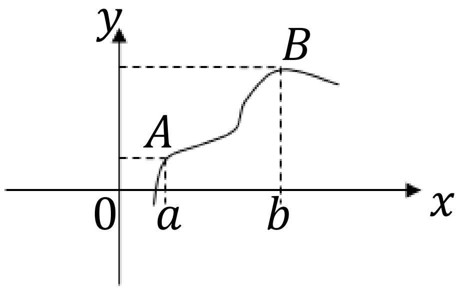


Fig. 11.4

Corollary 11.6. If the function $y = f(x)$ is continuous on the segment $[a, b]$ and takes values of different sign on its ends, then inside the segment $[a, b]$ there is at least one point C at which this function turns to zero $f(C) = 0, C \in (a,b)$.(fig. 11.5).

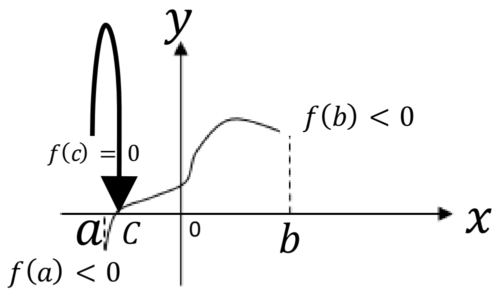


Fig. 11.5

Note that theorems 11.4 and 11.5 are not true for the function that suffers a break on the segment or the segment turns into the open interval.

§12. Derivative of the function.

12.1. Definition of the derivative, its mechanics and geometric meaning. The equation of tangent and normal to the curve.

Let the function $y = f(x)$ be defined on some interval $(a;b)$. We will perform the following actions:

let´s give the argument *x* an increment Δx: $x \in (a;b),\ x + \Delta x \in (a;b)$;

find the corresponding increment of the function $\Delta y = f(x + \Delta x) - f(x)$;

let´s make the ratio of the increment of the function to the increment of the argument.

If this limit exists, then it is called the derivative of the function $f(x)$ and it is denoted by one of the next symbols: $f'(x),\ f'_x,\ y',\ y'_x,\ \frac{dy}{dx}$.

The derivative of the function $y = f(x)$ at the point x_0 is called the limit of the ratio of the increment of the function to the increment of the argument, when the increment of the argument tends to zero. That is: $y' = \lim\limits_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ or $f'(x) = \lim\limits_{\Delta x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

A function $y = f(x)$ that has a derivative at each point in the interval $(a;b)$ is called differentiable on this interval. The operation of finding the derivative of a function is called differentiation. The value of the derivative of the function $y = f(x)$ at the point x_0 is denoted by one of the symbols: $f'(x_0), y'\big|_{x=x_0}$ or $y'(x_0)$.

Example 12.1. Find the derivative of the function $y = C,\ C = const$.

$$f(x) = C,\ f(x + \Delta x) = C \Rightarrow \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{C - C}{\Delta x} = 0 \Rightarrow \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

Thus, $C' = 0$.

Similarly, the derivatives of the basic elementary function could be find:

1. $(u^{\alpha})' = \alpha u^{\alpha - 1} \cdot u',\quad \alpha \neq 0$; (in particular, $\left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}} \cdot u'$; $\left(\frac{1}{u}\right)' = \frac{-1}{u^2} \cdot u'$);

2. $(a^u)' = a^u \cdot \ln a \cdot u'$;

3. $(e^u)' = e^u \cdot u'$;

4. $(\log_a u)' = \frac{1}{u \cdot \ln a} \cdot u'$;

5. $(\ln u)' = \frac{1}{u} \cdot u'$;

6. $(\sin u)' = \cos u \cdot u'$;

7. $(\cos u)' = -\sin u \cdot u'$;

8. $(\text{tg} u)' = \frac{1}{\cos^2 u} \cdot u'$;

9. $(\text{ctg} u)' = -\frac{1}{\sin^2 u} \cdot u'$;

10. $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$; $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$;

11. $(\text{arctg} u)' = \frac{1}{1+u^2} \cdot u'$; $(\text{arcctg} u)' = -\frac{1}{1+u^2} \cdot u'$;

12. $(\text{sh} u)' = \text{ch} u \cdot u'$;

13. $(\text{ch} u)' = \text{sh} u \cdot u'$;

14. $(\text{th} u)' = \frac{1}{\text{ch}^2 u} \cdot u'$; $(\text{cth} u)' = -\frac{1}{\text{sh}^2 u} \cdot u'$.

12.2. Mechanics meaning for the derivative of the function.

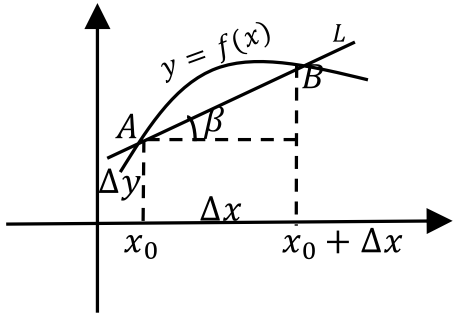
If s(*x*) is interpreted as a function that describes the dependence of the distance traveled on the time date, then

$\frac{\Delta s}{\Delta t}=\frac{s(t+\Delta t)-s(t)}{\Delta t}$ is interpreted as the average speed of the movement along the path traveled in time Δt. Then $\lim\limits_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=s'(t)$ is interpreted as the instantaneous speed of the movement at the given time *t*. Generalizing, we can say that if the function $y=f(x)$ describes a process, then the derivative is the speed of this process. This is the physical meaning of the derivative.

12.3. Geometric interpretation of the derivative.

On the graph of the function $y=f(x)$, we consider two points corresponding to the values of the arguments of the function $y=f(x)$: *x* and $x+\Delta x$. These are the points A and B. AB - secant to the function graph. Angle β - the angle of inclination of the secant L to the axis Ox. Then

$\mathrm{tg}\,\beta=\frac{\Delta y}{\Delta x}$. When $\Delta x\to 0$ the secant tends to occupy the position of the tangent.



In the limit, the secant has one point in common with the graph of the function at the point A. That is, the secant turns into the tangent to the graph of the function at the point x_0.

Thus, the derivative of the function $y=f(x)$ at the point *x* ($f'(x)$) is the tangent of the angle of inclination of the tangent to the graph of the function $y=f(x)$ at the point with coordinates $(x,\ f(x))$.

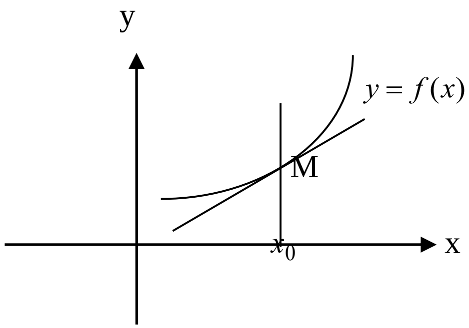
That is, the derivative of a function at a point *x* equals to the angular coefficient of the tangent to the graph of the function at a point k_{tang} whose abscissa is *x*. This is the geometric meaning of the derivative.

12.4. The equation of the tangent and normal lines to the curve.

If the point M has coordinates $M(x_0, y_0)$, then the angular coefficient of the tangent is $k = f'(x_0)$. Using the equation of a straight line passing through a given point x_0

($y - y_0 = k(x - x_0)$), we can write the equation of the tangent: $y - y_0 = f'(x_0)(x - x_0)$.

A straight line perpendicular to the tangent at the point of contact is called a normal line for the curve.



Since the normal is perpendicular to the tangent, its angular coefficient $k_{norm} = -\frac{1}{k_{\text{tang}}} = -\frac{1}{f'(x_0)}$. Therefore, the normal equation has the form

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \text{ (if } f'(x_0) \neq 0\text{).}$$

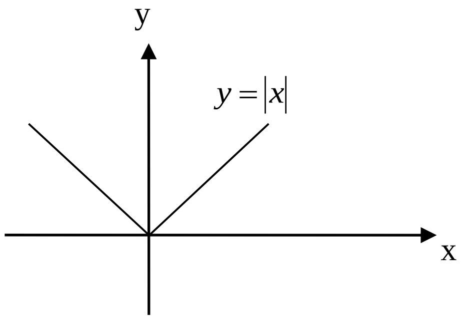
12.5. Relationship between continuity and differentiability of a function.

Theorem 12. 1. If a function is differentiable at a point, then it is continuous at that point.

Proof. Let the function $f(x)$ be differentiable at some point *x*. Hence there is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$. Accoding to Theorem 17.5 on the relation of a function, its limit, and an infinitesimal function, we have $\frac{\Delta y}{\Delta x} = f'(x) + \alpha$, where $\alpha \to 0$ at $\Delta x \to 0$, that is $\Delta y = f'(x)\Delta x + \alpha \cdot \Delta x$. Passing to the limit at $\Delta x \to 0$, we get $\lim_{\Delta x \to 0} \Delta y = 0$ and this means that the function $y = f(x)$ is continuous at the point *x*.

The inverse theorem is incorrect: a continuous function may not have a derivative. For example a function $y = |x|$.

$$y = |x| = \begin{cases} x, \ x \ge 0 \\ -x, \ x < 0. \end{cases}$$



The function under consideration is continuous at the point $x = 0$, but it is not differentiable at this point.

Really, at the point $x = 0$ we have

$$\frac{\Delta y}{\Delta x} = \frac{f(0+\Delta x) - f(0)}{\Delta x} = \frac{|0+\Delta x| - |0|}{\Delta x} = \frac{|\Delta x|}{\Delta x} = \begin{cases} 1, \ \Delta x \ge 0 \\ -1, \ \Delta x < 0 \end{cases}.$$

That is, $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ does not exist and the function $y=|x|$ doesn´t have a derivative at this point $x=0$, a graph of function doesn´t have a tangent at this point.

Comments:1. There are one-way limits for the function $y=|x|$ at the point $x=0$

$\lim\limits_{\Delta x\to 0-0}\dfrac{\Delta y}{\Delta x}=-1,\ \lim\limits_{\Delta x\to 0+0}\dfrac{\Delta y}{\Delta x}=1$. In such case, the function is said to have one-side derivatives and we denote it by $f_-'(x)$ and $f_+'(x)$. If $f_-'(x)\neq f_+'(x)$, then a derivative doesn´t exist at the point *x*. There is no derivative at the point of discontinuity of the function.

2. The derivative of a continuous function itself is not nessarily a non-continuous function. If a function $y=f(x)$ has a continuous derivative $y'=f'(x)$ in some interval $(a;b)$ then the function is called smooth.

12.6. The derivative of the sum, difference, product, quotient of two function.

Finding derivatives directly by definition in practice is associated with certain difficulties. Therefore, the following rules are used, which simplify the task of finding derivatives. Let $u=u(x)$ and $v=v(x)$ be two differentiable function in a certain interval $(a;b)$.

Theorem 12.2. The derivative of the sum(difference) for two functions is equal to the sum(difference) of the derivatives of these functions $(u\pm v)'=u'\pm v'$.

Proof. Denoty by $y=u\pm v$. By the definition of the derivative and the main theorems about the limits, we obtain:

$$y'=\lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x)\pm v(x+\Delta x)-(u(x)\pm v(x))}{\Delta x}\right)==\lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x)-u(x)}{\Delta x}\pm\frac{v(x+\Delta x)-v(x)}{\Delta x}\right)=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\pm\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=u'\pm v'.$$

That is: $(u\pm v)'=u'\pm v'$.

The theorem holds for any number terms.

Theorem 12.3. The derivative of the product of two functions is equal to the product of the derivative of the first factor by the second factor, plus the product of the first factor by the derivative of the second factor. $(u\cdot v)'=u'v+v'u$

Proof. Let $y=u\cdot v$. Then

$$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x)\cdot v(x+\Delta x) - u(x)\cdot v(x)}{\Delta x}\right) =$$

$$= \lim_{\Delta x\to 0}\left(\frac{[u(x+\Delta x)\cdot v(x+\Delta x) - (u(x)\cdot v(x+\Delta x)] + [u(x)\cdot v(x+\Delta x) - u(x)\cdot v(x)]}{\Delta x}\right) =$$

$$= \lim_{\Delta x\to 0}\frac{v(x+\Delta x)\cdot(u(x+\Delta x)-u(x))}{\Delta x} + \lim_{\Delta x\to 0}\frac{u(x)[v(x+\Delta x)-v(x)]}{\Delta x} =$$

$$= \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\cdot v(x+\Delta x) + \lim_{\Delta x\to 0} u(x)\frac{\Delta v}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\cdot\lim_{\Delta x\to 0} v(x+\Delta x) + \lim_{\Delta x\to 0} u(x)\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} =$$

$= u'(x)v(x) + u(x)v'(x)$. That is: $(u\cdot v)' = u'v + v'u$.

It can be shown that

$(Cu)' = Cu'$, C=const

$(u\cdot v\cdot \omega)' = u'uv + v'u\omega + uv\omega'$.

Theorem 12.4. The derivative of the quotient of two functions $u(x)$ and $v(x)$ ($v(x)\neq 0$) is a fraction where the numerator is the derivative of the numerator of the original fraction multiplied by the denumerator for the original fraction minus the derivative of the denominator of the original fraction multiplied by the numerator of the original fraction. The denominator of the nesulting fraction is the square of the denominator for the original fraction.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, (v\neq 0).$$

Proof. Let $y = \frac{u}{v}$. Then

$$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{u(x+\Delta x)-u(x)+u(x)}{v(x+\Delta x)-v(x)+v(x)} - \frac{u(x)}{v(x)}}{\Delta x} =$$

$$\lim_{\Delta x\to 0}\frac{\dfrac{u(x)+\Delta u}{v(x)+\Delta v}-\dfrac{u(x)}{v(x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{u(x)v(x)+\Delta u\cdot v(x)-u(x)v(x)+\Delta v\cdot u(x)}{\Delta x(v(x)+\Delta v)\cdot v(x)}=$$

$$=\lim_{\Delta x\to 0}\frac{\Delta u\cdot v(x)-\Delta v\cdot u(x)}{\Delta x(v(x)+\Delta v)\cdot v(x)}=\lim_{\Delta x\to 0}\frac{\dfrac{\Delta u}{\Delta x}\cdot v(x)-\dfrac{\Delta v}{\Delta x}\cdot u(x)}{v^2(x)+\Delta v\cdot v(x)}=$$

$$=\frac{v\lim\limits_{\Delta x\to 0}\dfrac{\Delta u}{\Delta x}-u\lim\limits_{\Delta x\to 0}\dfrac{\Delta v}{\Delta x}}{v^2+v\lim\limits_{\Delta x\to 0}v}=\frac{vu'-uv'}{v^2}.$$

That is $\left(\dfrac{u}{v}\right)'=\dfrac{vu'-uv'}{v^2}$.

Investigation 12.1. $\left(\dfrac{C}{v}\right)'=C\left(\dfrac{1}{v}\right)'=C\left[\dfrac{1'\cdot v-v'\cdot 1}{v^2}\right]=C\dfrac{(-v')}{v^2}=-C\dfrac{v'}{v^2}$, C=const.

12.7. The derivative of a composition functions.

Let $y=f(u)$ and $u=\varphi(x)$, than $y=f(\varphi(x))$ is a complex function with an intermediate argument *u* and an independent argument *x*.

Theorem 12.5.(chain rule) If the function $u=\varphi(x)$ has a derivative u'_x at the point *x*, and the function $y=f(u)$ has a derivative at the corresponding point $u=\varphi(x)$, then the composed function $y=f(\varphi(x))$ has a derivative y'_x at the point *x*, which is found by the formula $y'_x=y'_u\cdot u'_x$.

Proof. By condition $\lim\limits_{\Delta u\to 0}\dfrac{\Delta y}{\Delta u}=y'_u\Leftrightarrow\Delta y=y'_u\cdot\Delta u+\alpha\Delta u$, (12.6)

where $\alpha\to 0$ at $\Delta u\to 0$.

The function $u=\varphi(x)$ has a derivative at the point *x*: $\lim\limits_{\Delta x\to 0}\dfrac{\Delta u}{\Delta x}=u'_x\Leftrightarrow\Delta u=u'_x\cdot\Delta x+\beta\Delta x$, where $\beta\to 0$ at $\Delta x\to 0$.

Substituting the values of Δu into equality (12.6), we get

$\Delta y = y'_u(u'_x\Delta x + \beta\Delta x) + \alpha(u'_x\Delta x + \beta\Delta x)$. That is

$\Delta y = y'_u u'_x \Delta x + y'_u \beta\Delta x + u'_x \alpha\Delta x + \beta\alpha\Delta x$.

$\frac{\Delta y}{\Delta x} = y'_u u'_x + y'_u\beta + u'_x\alpha + \beta\alpha$, then at $\Delta x \to 0$.

$y'_x = y'_u u'_x$ ($y'_u\beta \underset{\Delta x\to 0}{\to} 0$, $u'_x\alpha \underset{\Delta x\to 0}{\to} 0$, $\alpha\beta \underset{\Delta x\to 0}{\to} 0$).

Thus, to find the derivative of a complex function $y = f(u) = f(\varphi(x))$, the derivative of this function with the intermediate argument *u* must be multiplied by the derivative of intermediate function u with the independent argument *x*. $y'_x = y'_u \cdot u'_x$.

This rule remain in effect if there are several intermediate arguments.

If $y = f(u)$, $u = \varphi(v)$, $v = g(x)$, then

$y'_x = y'_u \cdot u'_v \cdot v'_x = f'_u \cdot \varphi'_v \cdot g'_x$.

Let $y = f(x)$ and $x = \varphi(y)$ be mutually inverse functions.

Theorem 12.6. If the function $y = f(x)$ is strictly monotone on the interval $(a;b)$ and has an unequal zero derivative at any point of this interval, then the inverse function $x = \varphi(y)$ also has a derivative at the corresponding point of the same interval, defined by the equality $\varphi'_y = \frac{1}{f'_x}$ or $x'_y = \frac{1}{y'_x}$.

Consider the inverse function $x = \varphi(y)$.

$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}$, if $\Delta y \to 0$. If $\Delta y \to 0$ then by virtue of the continuity of the inverse function $\Delta x \to 0$. And since $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = f'(x) \neq 0$, then from (12.7)

follows the equality

$\lim\limits_{\Delta y\to 0}\dfrac{\Delta x}{\Delta y}=\dfrac{1}{\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}}=\dfrac{1}{f'(x)}=\varphi'(y)$. That is $\varphi'_y=\dfrac{1}{f'_x}$. Thus, the derivative of the inverse function is equal to the inverse of the derivative of the given function.

Example 12.3. Find the derivative of the function:

$$y=\log_2{}^3\operatorname{tg}x^4=\left(\log_2\operatorname{tg}x^4\right)^3=u^3$$

$$u=\log_2 z,\ z=\operatorname{tg}q,\ q=x^4,\ y'_x=y'_u\cdot u'_z\cdot z'_g\cdot g'_x$$

$$y'=3\left(\log_2\operatorname{tg}x^4\right)^2\cdot\left(\log_2\operatorname{tg}x^4\right)'_x=3\left(\log_2\operatorname{tg}x^4\right)^2\cdot\frac{1}{\ln 2}\cdot\frac{1}{\operatorname{tg}x^4}\left(\operatorname{tg}x^4\right)'_x=$$

$$=3\left(\log_2\operatorname{tg}x^4\right)^2\cdot\frac{1}{\ln 2}\cdot\frac{1}{\operatorname{tg}x^4}\cdot\frac{1}{\cos^2 x^4}\left(x^4\right)'==3\log^2{}_2\operatorname{tg}x^4\cdot\frac{1}{\ln 2}\cdot\frac{1}{\operatorname{tg}x^4}\cdot\frac{4\cdot x^3}{\cos^2 x^4}=3\log^2{}_2\operatorname{tg}x^4\cdot\frac{1}{\ln 2\cdot\operatorname{tg}x^4}\cdot\frac{1}{\cos^2 x^4}\cdot 4x^3.$$

12.8. Table of derivatives.

The rules of differentiation, formulas of derivatives of basic elememtary functions are written in the form of a table.

Rules of differentiation.

1. $(u\pm v)'=u'\pm v'$
2. $(u\cdot v)'=u'v+v'u$, $(Cu)'=Cu'$ if C=const.
3. $\left(\dfrac{u}{v}\right)'=\dfrac{vu'-uv'}{v^2}$, $\left(\dfrac{C}{v}\right)'=-C\dfrac{v'}{v^2}$
4. $y'_x=y'_u\cdot u'_x$, if $y=f(u)$ and $u=\varphi(x)$
5. $x'_y=\dfrac{1}{y'_x}$, $y'_x=\dfrac{1}{x'_y}$, if $y=f(x)$ and $x=\varphi(y)$.

Differentiation formulas.

$C' = 0$; if C=const

$(u^{\alpha})' = \alpha u^{\alpha-1} \cdot u', \quad \alpha \neq 0;$, in particular $\left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}} \cdot u'; \quad \left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$

$(a^u)' = a^u \cdot \ln a \cdot u';$ in particular $(e^u)' = e^u \cdot u';$

$(\log_a u)' = \frac{1}{u \cdot \ln a} \cdot u';$ in particular $(\ln u)' = \frac{1}{u} \cdot u';$

$(\sin u)' = \cos u \cdot u';$

$(\cos u)' = -\sin u \cdot u';$

$(\mathrm{tg} u)' = \frac{1}{\cos^2 u} \cdot u';$

$(\mathrm{ctg}\, u)' = -\frac{1}{\sin^2 u} \cdot u';$

$(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u';$

$(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u';$

$(\mathrm{arctg} u)' = \frac{1}{1+u^2} \cdot u';$

$(\mathrm{arcctg} u)' = -\frac{1}{1+u^2} \cdot u';$

$(\mathrm{sh}\, u)' = \mathrm{ch}\, u \cdot u';$

$(\mathrm{ch}\, u)' = \mathrm{sh}\, u \cdot u';$

$$(\text{th}u)'=\frac{1}{\text{ch}^2 u}\cdot u';$$

$$(\text{cth}u)'=-\frac{1}{\text{sh}^2 u}\cdot u'.$$

Example 12.10. Find the derivative of the function:

$y=x^4-3x^3+2x-1$. Since $x'=1$, we have

$$y'=(x^4-3x^3+2x-1)'=(x^4)'-(3x^3)+(2x)'-1'=4x^3\cdot x'-3(x^3)'+2x'-0=$$

$$=4x^3-9x^2\cdot x'+2=4x^3-9x^2+2.$$

Example 12.11. $y=\dfrac{2x^3}{\text{tg}\,x}$.

$$y'=\left(\frac{2x^3}{\text{tg}\,x}\right)'=2\left(\frac{x^3}{\text{tg}\,x}\right)'=2\frac{(x^3)'\,\text{tg}\,x-x^3(\text{tg}\,x)'}{(\text{tg}\,x)^2}=$$

$$=2\frac{3x^2\cdot x'\,\text{tg}\,x-x^3\cdot\dfrac{1}{\cos^2 x}x'}{\text{tg}^2\,x}=2\frac{3x^2\,\text{tg}\,x-\dfrac{x^3}{\cos^2 x}}{\text{tg}^2\,x}.$$

Example 12.12. $y=\cos(\ln^{12}2x)$.

$$y'=\sin(\ln^{12}2x)\cdot 12\cdot\ln^{11}2x\cdot\frac{1}{2x}\cdot 2.$$

$y=\cos u,\ u=v^{12},\ v=\ln t,\ t=2x$. That is: $y'_x=y'_u\cdot u'_v\cdot v'_t\cdot t'_x$.

§13. Differentiation of implicit and parametric functions.

13.1. An implicity defined function.

If the function is given by the equation $y = f(x)$, resolved with respect to y, then the function is given explicity.

The implicit assignment of a function is understood as the assignment of a function in the form of the equation $F(x;\ y) = 0$, which is not resolved with respect to y.

Any explicity given function $y = f(x)$ can be written as implicity given finction by the equation $f(x) - y = 0$. The converse statement is not true. It is not always easy, and sometimes impossible, to solve the equation with respect to y ($y + 2x + \cos y - 1 = 0$ or $2^y - x + y = 0$).

For example, if an implicity function is given by the equation $F(x;\ y) = 0$, then to find the derivative of y with respect to x, it is not necessary to solve the equation with respect to y. It is sufficient to differentiate the equation with respect to x considering y as function of x.

Then the resulting equation is resolved with respect to y'.

As a result, the derivative of the implicit function is expressed in terms of the argument x and the function y.

Example 13.1. Find the derivative of the function y, given implicity by the equation $x^3 + y^3 - 3xy = 0$.

$$(x^3 + y^3 - 3xy)' = 0'$$

$$3x^2 + 3y^2y' - 3(x'y + xy') = 0$$

$$3x^2 + 3y^2y' - 3(y + xy') = 0 \quad x^2 + y^2y' - y - xy' = 0$$

$$x^2 - y = y'(x - y^2)$$

$$y' = \frac{x^2 - y}{x - y^2}.$$

13.2. A function defined parametrically.

Let the relationship between the argument x and the function y be given parametrically in the form of two equations

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad (13.1)$$

where t is anxiliary variable called the parameter.

$y = f_1(x)\cdot f_2(x)\cdot f_3(x)$. We prolog this equality using the properties of the logarithm.

$\ln y = \ln f_1(x) + \ln f_2(x) + \ln f_3(x)$.

Let us take the derivative of the total part of the obtained equality.

$$\frac{1}{y}y'_x = \frac{1}{f_1(x)} f_1'(x) + \frac{1}{f_2(x)} f_2'(x) + \frac{1}{f_3(x)} f_3'(x).$$

Let us solve this equation with respect to y'_x.

Or $y = f(x)^{\varphi(x)}$. To find the derivation of a function y'_x we prolog this equality using the properties of the logarithm.

$\ln y = \ln f(x)^{\varphi(x)} = \varphi(x)\ln f(x)$.

Let us take the derivative of the total part of the obtained equality.

$(\ln y)' = [\varphi(x)\ln f(x)]'$.

$$\frac{1}{y}y'_x = \varphi'(x)\ln f(x) + \varphi(x)\cdot\frac{1}{f(x)}\cdot f'(x).$$

Let us solve this equation with respect to y'_x.

$$y'_x = \left[\varphi'(x)\ln f(x) + \frac{\varphi(x) f'(x)}{f(x)}\right]\cdot f(x)^{\varphi(x)}.$$

Find the derivative of y'_x, assuming that the function $x(t)$ and $y(t)$ have derivatives and the function $x(t)$ has $x'(t) \neq 0$ on the entire domain of definition. Than

$$y'_x = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'_t}{x'_t} = y'_x . \tag{13.2}$$

The resulting formula allows us to find y'_x from a function given parametrically, without finding a direct dependence of y on x .

Example 13.1. Let $\begin{cases} x = t^3 \\ y = t^2 \end{cases}$. Find y'_x.

Solution. We have $x'_t = 3t^2, y'_t = 2t$. Therefore (13.2) $y'_x = \frac{2t}{3t^2} = \frac{2}{3t}$.

§14. Logarithmic differentiation.

In the case when a function is given as a product of several functions or as a function in the degree of a function, it is advisable to first prolog it to find the derivative of a function y.

Example 14.1. Find y'_x, if

$$y=\frac{(x^2+2)\sqrt[4]{(x-1)^3}\cdot e^x}{(x+5)^3}.$$

$$\ln y=\ln(x+2)+\ln(x+1)^{\frac{3}{4}}+\ln e^x-\ln(x+5)^3 \text{ or}$$

$$(\ln y)'=\left(\ln(x+2)+\frac{3}{4}\ln(x+1)+x-3\ln(x+5)\right)'$$

$$\frac{1}{y}y'=\frac{1}{x+2}+\frac{3}{4}\cdot\frac{1}{x+1}+1-3\cdot\frac{1}{x+5}.$$

Than $y'=\left[\frac{1}{x+2}+\frac{3}{4}\cdot\frac{1}{x+1}+1-\frac{3}{x+5}\right]\cdot y$ or

$$y'=\left[\frac{1}{x+2}+\frac{3}{4}\cdot\frac{1}{x+1}+1-\frac{3}{x+5}\right]\cdot\frac{(x^2+2)\sqrt[4]{(x-1)^3}\cdot e^x}{(x+5)^3}.$$

Example 14.2. Find y'_x, if $y=(\sin 2x)^{x^2+1}$.

$$\ln y=(x^2+1)\ln\sin 2x.$$

$$(\ln y)'=\left[(x^2+1)\ln\sin 2x\right]'$$

$$\frac{1}{y}y'_x=(x^2+1)'\ln\sin 2x+(x^2+1)(\ln\sin 2x)'$$

$$\frac{1}{y}y'_x = 2x\ln\sin 2x + (x^2+1)\frac{1}{\sin 2x}\cos 2x\cdot 2$$

$$y'_x = \left[2x\ln\sin 2x + 2(x^2+1)\,\mathrm{tg}\,2x\right]\cdot(\sin 2x)^{x^2+1}.$$

§15. Higher-order derivatives.

The derivative $y' = f'(x)$ of a function $y = f(x)$ is also a function of x and it is called a first-order derivative.

If the function $f'(x)$ is differentiable, then its derivative is called the derivative of the second-order derivative and is denoted by y'' (or $f''(x)$, $\frac{d^2y}{dx^2}$, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{dy'}{dy}$). So, $y'' = (y')'$.

The derivative of the second-order derivative, if it exists, is called the third-order derivative and is denoted by y''' (or $f'''(x)$, $\frac{d^3y}{dx^3}$, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$, $\frac{dy'}{d^2y}$).

So, $y''' = (y'')'$.

The derivative of the n-th order is called the derivative of the $(n-1)$-th order.

$y^{(n)} = (y^{(n-1)})'$.

Derivatives above the first order are called higher order derivatives.

Example 15.1. Find the derivative of the 13-th order of the function $y = \sin x$.

$$y' = (\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$y'' = (\cos x)' = -\sin x = \sin\left(x + \frac{\pi}{2}\cdot 2\right)$$

$$y''' = (-\sin x)' = -\cos x = \sin\left(x + \frac{\pi}{2}\cdot 3\right)$$

$$y^{(13)} = \sin\left(x + \frac{\pi}{2}\cdot 13\right).$$

15.1. Higher-order derivatives of implicitly defined functions.

Let the function $y = f(x)$ be given implicitly as an equation $F(x;y) = 0$. Differetiating this equation with respect to x and solving the resulting equation with respect to y', we find the first-order derivative.

Differentiating the first derivative by x, we get the second derivative of the implicit function. It will include x, y, and y'. Substituting the already found value of y' in the expression of the second derivative, we express y'' in terms of x and y. We do the same for finding the third-order derivative, and so on.

Example 15.2. Find y''', if $x^2 + y^2 = 1$.

Decision. We differentiate the equation by x.

We obtain $2x + 2yy' = 0$. Futher, $x = -yy'$, $y' = -\frac{x}{y}$

$$y'' = \left(-\frac{x}{y}\right)' = -\left(\frac{x'y - xy'}{y^2}\right) = \frac{xy' - y}{y^2} = \frac{x\left(-\frac{x}{y}\right) - y}{y^2} = \frac{-x^2 - y^2}{y^3} =$$

$$= \frac{-x^2 - 1 + x^2}{y^3} = \frac{-1}{y^3} \text{ (since } x^2 + y^2 = 1\text{)}.$$

$$y'' = \frac{-1}{y^3} = -y^{-3}.$$

$$y''' = -1 \cdot (-3) y^{-4} \cdot y' = \frac{3y'}{y^4} = \frac{3}{y^4}\left(\frac{-x}{y}\right) = \frac{-3x}{y^5}.$$

Thus, $y''' = \frac{-3x}{y^5}$.

15.2. Higher-order derivatives of functions defined parametrically.

Let the function $y = f(x)$ given parametrically. As you know, the first derivative of y' is found by the formula $y'_x = \dfrac{y'_t}{x'_t}$.

We find the second derivative y'' of the function given parametrically.

$$y''_x = (y'_x)' = \frac{dy'_x(t)}{dx} = \frac{dy'_x(t)}{dt} \cdot \frac{dt}{dx(t)} = \frac{\dfrac{dy'_x(t)}{dt}}{\dfrac{dx}{dt}} = \frac{\left(y'_x\right)'_t}{x'_t}.$$

Thus, $y''_x = \dfrac{\left(y'_x\right)'_t}{x'_t}$. Similary, $y'''_x = \dfrac{\left(y''_x\right)'_t}{x'_t}$.

Example 15.3. Find the second derivative of a function given parametrically

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}.$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{(\sin t)'}{(\cos t)'} = \frac{\cos t}{-\sin t} = -\mathrm{ctg}\, t$$

$$y''_x = \frac{(y'_t)'_t}{x'_t} = \frac{(-\mathrm{ctg}\, t)'_t}{-\sin t} = \frac{\dfrac{1}{\sin^2 t}}{-\sin t} = \frac{1}{\sin^3 t} = y''_x.$$

§16. The differential of the function.

16.1. The concept of the differential of a function.

Let the function $y = f(x)$ have a nonzero derivative at the point x. $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) \neq 0$. Then, by the connection theorem of a function, its limit and an infinitesimal function, we can write

$\frac{\Delta y}{\Delta x} = f'(x) + \alpha$, where $\alpha \to 0$ to $\Delta x \to 0$, or $\Delta y = f'(x)\Delta x + \alpha \cdot \Delta x$.

Thus, the increment of the function Δy is the sum of two terms $f'(x)\Delta x$ and $\alpha \cdot \Delta x$, which are infinitesimal at $\Delta x \to 0$.

In this case, the first term is an infinitesimal function of the same order of smallness with Δx, since $\lim\limits_{\Delta x \to 0} \frac{f'(x)\Delta x}{\Delta x} = f'(x) \neq 0$.

The second terms is an infinitesimal function of a higher order than Δx:

$$\lim_{\Delta x \to 0} \frac{\alpha \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \alpha = 0.$$

Therefore, the first term is called the main part of the increment of the function Δy.

The differential of a function $y = f(x)$ at the point x is the main part of its increment, equal to the product of the derivative of the function by the increment of the argument, and is denoted by dy (or $df(x)$)

$$dy = f'(x)\Delta x \qquad (16.1)$$

The differential dy is also called a first-order differential.

Find the differential of the independent variable x, that is, the differential of the function $y = x$. Since $y' = x' = 1$, then, according to formula (16.1), we have $dx = x'\Delta x = \Delta x$.

That is, the differential of the independent variable is equal to the increment of this variable. Therefore, the formula (16.1) can be written as:

$$dy = f'(x)dx. \qquad (16.2)$$

In other words, the differential of a function is equal to the product of the derivative of that function by the differential of the independent varable.

From formula (16.2) follows the equality $\frac{dy}{dx} = f'(x)$.

Now the notation of the derivative of the function can be considered as the ratio of the differentials dy and dx.

Example 16.1. Find the differential of the function $f(x) = 3x^2 - \sin(1+2x)$.

$$df(x) = f'(x)dx = \left[3x^2 - \sin(1+2x)\right]' dx = \left[6x - 2\cos(1+2x)\right]dx.$$

Example 16.2. Find the differential of the function:

$$y = \ln(1+e^{10x}) + \sqrt{x^2+1}.$$

$$dy = y'_x dx = \left[\ln(1+e^{10x}) + \sqrt{x^2+1}\right]' dx = \left[\frac{1}{1+e^{10x}} e^{10x} \cdot 10 + \frac{2x}{2\sqrt{x^2+1}}\right]dx.$$

Thus, $dy = \left[\frac{10e^{10x}}{1+e^{10x}} + \frac{x}{\sqrt{x^2+1}}\right]dx$.

16.3. Basic theorems about differentials.

The main theorems about differentials are easily obtained by using the relation of the differential and the derivative of the function($dy = f'(x)dx$).

The corresponding derivative theorems are also used. For example, since the derivative of the function $y = c$ is zero, the differential of the constant is zero.

$$dy = c'(x)dx = 0 \cdot dx = 0.$$

Theorem 16.1. The differential of the sum, difference, product and quotient of two differentable functions is defined by the following formulas:

$$d(u \pm v) = du \pm dv$$

$$d(uv) = udv + vdu$$

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2} \quad (v \neq 0).$$

We prove, for example, the second formula

$$d(uv) = (uv)'dx = (u'v + uv')dx = vu'dx + uv'dx = vdu + udv = udv + vdu.$$

Theorem 16.2. The differential of a complex function is equal to the product of the derivative of this function with respect to the intermediate argument and the differential of this intermediate argument.

Let $y = f(u)$ and $u = \varphi(x)$ be two differentiable functions forming a complex function. By the theorem on the derivative of a complex function, we have

$$y'_x = y'_u \cdot u'_x.$$

Multiply both parts of this equality by dx

$y'_x dx = y'_u \cdot u'_x dx$. We get $dy = f'_u du$.

Comparing formulas $dy = y'_x dx$ and $dy = y'_u du$, we see that the first differential of a function is determined by the same formula, regardless of whether its argument is an independent variable or a function of another argument. This property of the differential is called the invariance (immutability) for the form of the first differential.

Formulas $dy = y'_x dx$ and $dy = y'_u du$ are identical in appearance. But there is a fundamental difference between them. In the first formula x is an independent variable. Hence $dx = \Delta x$. In the second formula v is a function of x. Therefore, generally speaking, $du \neq \Delta u$. Using the definition of the differential and the basic theorems about differentials, it is easy to transform the table of derivatives into a table of differentials. For example,

$$d(\cos u) = (\cos u)' \cdot du = -\sin u du.$$

16.4. Table of differentials.

1. $d(u \pm v) = du \pm dv$
2. $d(uv) = udv + vdu$, in particular, $d(cu) = cdu$

3. $d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$ $(v\neq 0)$, in particular, $d\left(\frac{c}{v}\right)=-\frac{cdv}{v^2}$

4. $dy=y'_x dx$, if $y=f(x)$

5. $dy=y'_u du$, if $y=f(u)$ and $u=\varphi(x)$

6. $dc=0$

7. $d(u^{\alpha})=\alpha\cdot u^{\alpha-1}du$

8. $d(a^u)=a^u\ln a\cdot du$, in particular, $d(e^u)=e^u\cdot du$.

9. $d(\log_a u)=\frac{1}{u\ln a}du$, in particular, $d(\ln u)=e^u\cdot du$

10. $d(\sin u)=\cos u du$

11. $d(\cos u)=-\sin u du$

12. $d(\mathrm{tg} u)=\frac{1}{\cos^2 u}du$

13. $d(\mathrm{ctg})=-\frac{1}{\sin^2 u}du$

14. $d(\arcsin u)=\frac{1}{\sqrt{1-u^2}}du$

15. $d(\arccos u)=-\frac{1}{\sqrt{1-u^2}}du$

16. $d(\mathrm{arctg}u) = \frac{1}{1+u^2}du$

17. $d(\mathrm{arcctg}u) = -\frac{1}{1+u^2}du$

18. $d(\mathrm{sh}u) = \mathrm{ch}udu$

19. $d(\mathrm{ch}u) = \mathrm{sh}udu$

20. $d(\mathrm{th}u) = \frac{1}{\mathrm{ch}^2 u}du$

21. $d(\mathrm{cth}u) = -\frac{1}{\mathrm{sh}^2 u}du$.

16.5. Applying the differential to approximate calculations.

It is known that $f'(x) = \lim\limits_{\Delta x \to 0}\frac{\Delta y}{\Delta x}$. Therefore, $\Delta y \approx f'(x)\Delta x$ or $f(x+\Delta x) - f(x) = f'(x)\Delta x$,

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x \qquad (16.3)$$

Formula (16.3) is used to calculate the approximate values of a function $f(x)$ at a subsequent point $x+\Delta x$, knowing the value of the function itself and its derivative at the previous point x .

The accuracy of the formula (16.3) increases with decreasing Δx .

Example 16.3. Calculate approximately $\mathrm{arctg}1{,}05$.

Consider the function $f(x) = \mathrm{arctg}x$. By the formula (16.3),

$\operatorname{arctg}(x+\Delta x)\approx \operatorname{arctg}x+(\operatorname{arctg}x)'\Delta x$. That is

$$\operatorname{arctg}(x+\Delta x)\approx \operatorname{arctg}x+\frac{1}{1+x^2}\Delta x.$$

Since 1,05=1+0,5, then $x=1,\ \Delta x=0{,}5$ we get $\operatorname{arctg}1{,}05=\operatorname{arctg}1+\frac{0{,}05}{1+1^2}=\frac{\pi}{4}+0{,}025\approx 0{,}810$.

It can be shown that the absolute error of formula (16.3) does not exceed the value of $M\cdot(\Delta x)^2$, where M – is the largest value of $|f''(x)|$ on the segment $[x, x+\Delta x]$. (See 17.2).

Example 16.4.

What path will the body take in free fall on the moon in 10,04 c from the start of the fall. The equation of free fall of a body $H=\frac{9m\cdot t^2}{2}$, $9m=1{,}6\,m/c^2$.

Solution. You need to find $H(10{,}04)$. Let´us use the approximate formula ($\Delta H\approx dH$)

$$H(t+\Delta t)\approx H(t)+H'(t)\Delta t.$$

At $t=10c,\ \Delta t=dt=0{,}04c\ \ H'(t)=9m\cdot t$ we find

$$H(10{,}04)\approx\frac{1{,}6\cdot 100}{2}+1{,}6\cdot 10\cdot 0{,}04=80+0{,}64=80{,}64\,(\text{m}).$$

A task for an independent solution.

A body of mass m=20 kg moves at a speed of $v=10{,}02\ \ m/c$. Calculate approximately the kinetic energy of the body $\left(E_k=\frac{mv^2}{2},\ E_k(10{,}02)\approx 1004(Dg)\right)$.

16.6. Differentials of higher orders.

Let $y = f(x)$ be a differentiable function, and its argument *x* is an independent variable.

Then its first differential $dy = f'(x)dx$ is function of x, one cane find the differential of this function. The differential from the differential of function $y = f(x)$ is called the second differential or the second-order differential and is denoted d^2y or $d^2f(x)$.

So, by definition $d^2y = d(dy)$. Find the expression of the second differential of the function $y = f(x)$. Since $dx = \Delta x$ and it does not depend on *x*, then when differentiating, we consider dx to be constant:

$$d^2y = d(dy) = d(f'(x)dx) = (f'(x)dx)'dx = f''(x)dxdx = f''(x)(dx)^2 \equiv f''(x)dx^2. ((dx)^2 \equiv dx^2).$$

Similarly, the third-order differential is determined and found. In general, a n-order differential is a (n-1) order differential

$d^n y = d(d^{n-1}y) = f^{(n)}(x)(dx)^n = f^{(n)}(x)dx^n$. Hence we find that

$$f'(x) = \frac{dy}{dx},\ f''(x) = \frac{d^2y}{dx^2},\ f'''(x) = \frac{d^3y}{dx^3}. \qquad (16.4)$$

That is, the derivative of a function can be considered as the ratio of its differential of the corresponding order to the corresponding degree of the differential of an independent variable.

Note that all the about formulas are valid only if *x* is an independent variable.

If (16.4) $y = f(x)$, where *x* is a function of some other independent variable, then the differential of the second and higher order not have the invariance property and is calculated using other formulas. Let´s show this by the example of a second-order differential.

Using the product differentiation formula $d(u \cdot v) = udv + vdu$, we obtain

$$d^2y = d(f'(x)dx) = d(f'(x))'dx + f'(x)d(dx) = f''(x)dxdx + f'(x)d^2x \Leftrightarrow$$

$$\Leftrightarrow d^2y = f''(x)dx^2 + f'(x)d^2x. \qquad (16.5)$$

Comparing formulas (16.4) and (16.5), we make sure that the case of a complex function, the formula of the second-order differential changes: the second term appears $f'(x)d^2x$. It is clear that if x is an independent variable, then

$d^2x = d(dx) = d(1\cdot dx) = dx\cdot d1 = dx\cdot 0 = 0$ and formula (16.5) go as (16.4).

Example 16.5. Find d^2y, if $y = e^{3x}$ and *x* is an independent variable.

Solution. $y' = 3e^{3x},\ y'' = 9e^{3x} \Rightarrow d^2y = f''(x)dx^2 \Leftrightarrow d^2y = 9e^{3x}dx^2$.

Example 16.6. Find d^2y, if $y = x^2$ and $x = t^3 + 1$ and *t* is an independent variable.

We have: $y' = 2x,\ y'' = 2,\ dx = 3t^2dt,\ d^2x = 6tdt^2$, than

$$d^2y = 2dx^2 + 2x6tdt^2 = 2(3t^2dt)^2 + 2(t^3+1)6tdt^2 = 18t^4dt^2 + 12t^4dt^2 + 12tdt^2 =$$

$$= (30t^4 + 12t)dt^2.$$

Other solution: $y = x^2$, $x = t^3 + 1$. Therefore, $y = (t^3+1)^2$. Further,

$d^2y = y''dt^2 = (30t^4 + 12t)dt^2$.

§17. The study of functions using derivatives.

17.1. Some theorems on differentable functions.

Let us consider a number of theorems of great theoretical and applied importance.

Theorem 17.1. (The Roll´s Theorem).

If the function $f(x)$ is continuous on the segment [a, b], differentiable on the interval (a, b) and the ends of the segment take the same values, then there will be at least on point $c \in (a, b)$ at which the derivative turn to zero, that is $f'(c) = 0$.

Theorem 17.2. (Cauchy´s Theorem).

If the functions $f(x)$ and $\varphi(x)$ are continuous on the segment [a, b], differentiable on the interval (a, b) and $\varphi(x) \neq 0\ \forall x \in (a, b)$, then there is at least one point $c \in (a, b)$ that the equality holds

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(c)}{\varphi'(c)}.$$

Theorem 17.3. (Lagrange´s Theorem).

If the function $f(x)$ is continuous on the segment [a, b], differentiable on the interval (a, b), then there is at least one point $c \in (a, b)$ such that the next equality holds: $f(b) - f(a) = f'(c)(b - a)$.

This formula is called the Lagrange formula or the formula of finite increments: the increment of the differentiable function on the segment [a, b] is equal to the increment of the argument multiplied by the value of the derivative of the function at some inner point of this segment.

Corollary 1. If the derivative of a function iz zero on a certain interval, then the function is constant on this interval.

Corollary 2. If two function have equal derivatives on a certain interval, then they differ from each other by a constant term.

17.2. The Lopitale rule.

Let´s consider a method of disclosing uncertainties of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$, which is based on the use of derivatives.

Theorem 17.4.(Lopital´s rule of disclosure of uncertainties of the form $\frac{0}{0}$).

Let the functions $f(x)$ and $\varphi(x)$ be continuous and differentiable in the neighborhood of the point x_0 and $f(x_0)=\varphi(x_0)=0$. If there is a

$\lim\limits_{x\to x_0}\frac{f'(x)}{\varphi'(x)}=l$, then $\lim\limits_{x\to x_0}\frac{f(x)}{\varphi(x)}=\lim\limits_{x\to x_0}\frac{f'(x)}{\varphi'(x)}=l$.

Example. 17.1. Find $\lim\limits_{x\to1}\frac{x-1}{x\ln x}=\left[\frac{0}{0}\right]=\lim\limits_{x\to1}\frac{(x-1)'}{(x\ln x)'}=\lim\limits_{x\to1}\frac{1}{\ln x+x\frac{1}{x}}=\lim\limits_{x\to1}\frac{1}{0+1}=\frac{1}{1}=1.$

$$\lim_{x\to0}\frac{1-\cos6x}{2x^2}=\left[\frac{0}{0}\right]=\lim_{x\to0}\frac{(1-\cos6x)'}{(2x^2)'}=\lim_{x\to0}\frac{6\sin6x}{4x}=\left[\frac{0}{0}\right]=\lim_{x\to0}\frac{6\cdot6\cos x}{4}=\lim_{x\to0}9\cos x=9.$$

Theorem 17.5.(Lopital´s rule of disclosure of uncertainties of the form $\frac{\infty}{\infty}$).

Let the functions $f(x)$ and $\varphi(x)$ be continuous and differentiable in the neighborhood of the point x_0 and $\lim\limits_{x\to x_0}f(x)=\lim\limits_{x\to x_0}\varphi(x)=\infty$, $\varphi'(x)\neq0$. If there is a $\lim\limits_{x\to\infty}\frac{f'(x)}{\varphi'(x)}=l$, then $\lim\limits_{x\to\infty}\frac{f(x)}{\varphi(x)}=\lim\limits_{x\to\infty}\frac{f'(x)}{\varphi'(x)}$.

Example.17.2. Find the limit.

$$\lim_{x\to\frac{\pi}{2}}\frac{\operatorname{tg}3x}{\operatorname{tg}5x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\frac{\pi}{2}}\frac{(\operatorname{tg}3x)'}{(\operatorname{tg}5x)'}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\cos^2 3x}\cdot3}{\frac{1}{\cos^2 5x}\cdot5}=\frac{3}{5}\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 5x}{\cos^2 3x}=\frac{3}{5}\lim_{x\to\frac{\pi}{2}}\frac{1+\cos10x}{1+\cos6x}=$$

$$=\left[\frac{0}{0}\right]=\frac{3}{5}\lim_{x\to\frac{\pi}{2}}\frac{-10\sin10x}{-6\sin6x}=\frac{3}{5}\cdot\frac{10}{6}\lim_{x\to\frac{\pi}{2}}\frac{\sin10x}{\sin6x}=\left[\frac{0}{0}\right]=\lim_{x\to\frac{\pi}{2}}\frac{10\cos10x}{6\cos6x}=\frac{5}{3}\lim_{x\to\frac{\pi}{2}}\frac{\cos10x}{\cos6x}=\frac{5}{3}.$$

$$\left(\cos\frac{10\pi}{2}=\cos5\pi=-1;\ \cos\frac{6\pi}{2}=\cos3\pi=\cos\pi=-1\right).$$

Another way.

$$\lim_{x\to\frac{\pi}{2}}\frac{\operatorname{tg}3x}{\operatorname{tg}5x}=\left[\frac{\infty}{\infty}\right]=\left|\begin{aligned}x-\frac{\pi}{2}=t\\ x=\frac{\pi}{2}+t\end{aligned}\right|=$$

$$=\lim_{t\to 0}\frac{\operatorname{tg}3\left(\frac{\pi}{2}+t\right)}{\operatorname{tg}5\left(\frac{\pi}{2}+t\right)}=\lim_{t\to 0}\frac{\operatorname{tg}\left(\frac{3\pi}{2}+3t\right)}{\operatorname{tg}\left(\frac{5\pi}{2}+5t\right)}=\lim_{t\to 0}\frac{\operatorname{tg}\left(\frac{\pi}{2}+3t\right)}{\operatorname{tg}\left(\frac{\pi}{2}+5t\right)}=$$

(we take into account the frequency of the function)

$$=\lim_{t\to 0}\frac{-\operatorname{ctg}3x}{-\operatorname{ctg}5x}=\lim_{t\to 0}\frac{\operatorname{tg}5x}{\operatorname{tg}3x}=\left[\frac{0}{0}\right]=\lim_{t\to 0}\frac{\frac{1}{\cos^2 5t}\cdot 5}{\frac{1}{\cos^2 3t}\cdot 3}=\frac{5}{3}.$$

17.3. Disclosure of various types of indeterminate form.

The Lopital´s rule is used to dislose the indeterminate form of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$, wich are called the main ones. The indeterminate form of type $\infty-\infty, 1^{\infty}, \infty^{0}, 0^{0}$ are reduced to two main types by identical transformations.

1. Let $\lim_{x\to x_0} f(x)=0$ and $\lim_{x\to x_0}\varphi(x)=\infty$. Then the following equivalent transformations are obvious:

$$\lim_{x\to x_0} f(x)\varphi(x)=(0\cdot\infty)=\lim_{x\to x_0}\frac{f(x)}{\frac{1}{\varphi(x)}}=\left[\frac{0}{0}\right](\text{or }\lim_{x\to x_0}\frac{\varphi(x)}{\frac{1}{f(x)}}=\left[\frac{\infty}{\infty}\right]).$$

Example.17.3.

$$\lim_{x\to 2}\operatorname{tg}\frac{\pi x}{4}(2-x)=[\infty\cdot 0]=\lim_{x\to 2}\frac{2-x}{\operatorname{ctg}\frac{\pi x}{4}}=\left[\frac{0}{0}\right]=\lim_{x\to 2}\frac{-1}{\frac{-1}{\sin^2\frac{\pi x}{4}}\cdot\frac{\pi}{4}}=\lim_{x\to 2}\frac{\sin^2\frac{\pi x}{4}}{\frac{\pi}{4}}=\frac{4}{\pi}.$$

2. Let $\lim\limits_{x\to x_0} f(x)=\infty$ and $\lim\limits_{x\to x_0}\varphi(x)=\infty$. Then you can do this

$$\lim_{x\to x_0}(f(x)-\varphi(x))=[\infty-\infty]=\lim_{x\to x_0}\left[\frac{1}{\frac{1}{f(x)}}-\frac{1}{\frac{1}{\varphi(x)}}\right]=\lim_{x\to x_0}\frac{\frac{1}{\varphi(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)}\cdot\frac{1}{\varphi(x)}}=\left[\frac{0}{0}\right].$$

Example.17.4.

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=[\infty-\infty]=\lim_{x\to 1}\frac{x-1-\ln x}{\ln x\cdot(x-1)}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}}=\frac{1}{2}.$$

3. Let $\lim\limits_{x\to x_0} f(x)=1$, $\lim\limits_{x\to x_0}\varphi(x)=0$ or $\lim\limits_{x\to x_0} f(x)=\infty$, $\lim\limits_{x\to x_0}\varphi(x)=0$, or $\lim\limits_{x\to x_0} f(x)=0$, $\lim\limits_{x\to x_0}\varphi(x)=0$. To find the limit of the form $f(x)^{\varphi(x)}$, it is convenient to first prolog the expression

$A=f(x)^{\varphi(x)}$.

Example. 17.5. Find $\lim\limits_{x\to 0}(\cos 2x)^{\frac{1}{x^2}}$.

We have the ucertainty of the form $[1^\infty]$. Prologarithm the expression

$A=(\cos 2x)^{\frac{1}{x^2}}$, $\ln A=\frac{1}{x^2}\ln\cos 2x$. Then

$$\lim_{x\to 0}\ln A=\lim_{x\to 0}\frac{\ln\cos 2x}{x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\frac{-2\sin 2x}{\cos 2x}}{2x}=-2\lim_{x\to 0}\frac{\operatorname{tg}2x}{2x}=-2\,.$$

That is $\ln\lim_{x\to 0}A=-2$. Therefore, $\lim_{x\to 0}A=e^{-2}$ and $\lim_{x\to 0}(\cos 2x)^{\frac{1}{x^2}}=e^{-2}$.

The solution can be made shorter if you use the basic logarithmic identity $f^{\varphi}=e^{\ln f^{\varphi}}$.

Then $\lim_{x\to 0}(\cos 2x)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to 0}\ln(\cos 2x)^{\frac{1}{x^2}}}=e^{-2}$.

Example.17.6. Find $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\operatorname{tg}x}=\left[\infty^0\right]$.

Solution.

$$\lim_{x\to 0}\left(\frac{1}{x}\right)^{\operatorname{tg}x}=\left[\infty^0\right]=\exp\left(\lim_{x\to 0}\operatorname{tg}x\cdot\ln\frac{1}{x}\right)=\exp\left(\lim_{x\to 0}\frac{\ln\frac{1}{x}}{\operatorname{ctg}x}\right)=\exp\left(\lim_{x\to 0}\frac{x\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{\sin^2 x}}\right)=$$

$$=\exp\left(\lim_{x\to 0}x\left(\frac{\sin x}{x}\right)^2\right)=e^{0\cdot 1}=e^0=1.$$

17.3. Increasing and decreasing functions.

One of the applications of the derivative is its application to the study of functions and the construction of a function graph.

We list the necessary and sufficient conditions for the increase and decrease of the function.

Theorem 17.6. (necessary conditions). If the function differentiable on the interval $(a; b)$ increases (decreases), then $f'(x) \geq 0$ ($f'(x) \leq 0$) for $\forall\ x \in (a; b)$.

Theorem 17.7. (sufficient conditions). If the function differentiable by $(a; b)$ and $f'(x) > 0$ ($f'(x) < 0$) for all $x \in (a; b)$, then this function increases (decreases) by the interval $(a; b)$.

Example.17.7. Investigate the function for increasing and decreasing: $f(x) = x^3 - 3x - 4$.

The function is defined on the entire numeric axis.

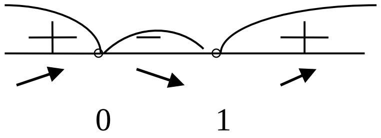


Fig. 17.1

$f'(x) = 3x^2 - 3x = 3x(x-1) > 0$.

Therefore(consequently), this function increases at intervals $(-\infty, 0) \cup (1, \infty)$ and decreases at interval (0, 1).

17.4. Maximum and minimum functions.

The point x_0 is called the maximum point of the function $y = f(x)$, if there is such δ - neighborhood of the point x_0, that for all $x \neq x_0$ from this neighborhood the inequality $f(x) < f(x_0)$ holds. The minimum point of the function is determined similary: x_0 - the minimum point of the function, if $\exists\delta\ \forall x: 0 < |x - x_0| < \delta \Rightarrow f(x) > f(x_0)$.

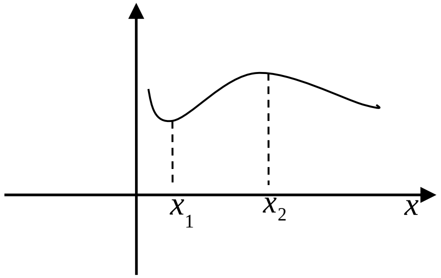


Fig. 17.2.

On fig.52.2 x_1 is the minimum point, x_2 is the maximum point. The value of the function at the maximum(minimum) point is called the maximum and the minimum of the function. The maximum(minimum) of the function is called the extremum of the function.

Consider the conditions for the existence of an extremum of the function.

Theorem 17.8. (necessary extremum conditions). If a differentiable function $f(x)$ has an extremum at a point x_0, then its derivative at this point is equal to zero: $f'(x_0) = 0$.

Let, for definiteness, x_0 be the maximum point. This means that the inequality $f(x_0) > f(x_0 + \Delta x)$ holds in the vicinity of the point x_0.

But then $\frac{\Delta y}{\Delta x}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}<0$, if $\Delta x>0$, and $\frac{\Delta y}{\Delta x}>0$, if $\Delta x<0$.

By the hypothesis of the theorem the derivative $f'(x_0)=\lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ exists.

Passing to the limit when $\Delta x\to 0$, we get $f'(x)\geq 0$, if $\Delta x<0$ and $f'(x_0)\leq 0$, if $\Delta x>0$.

That´s why $f'(x_0)=0$.

The assertion of the theorem is proved similarly if x_0 is the minimum point of the function.

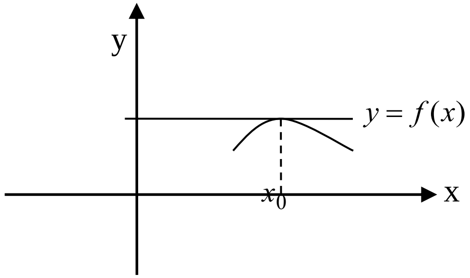


Fig. 17.3.

Geomrtrically, the equality $f'(x_0)=0$ means, that at the extremum point of a differentiable function, the tangent to its graph is parollel to the Ox axis.

Note, that the converse theorem is not true. If $f'(x_0)=0$, then this does not mean that x_0 is an extremum point.

Example. 17.8. For a function $y=x^3$ its derivative $y'=3x^2$ is zero at the point $x_0=0$, but the point $x_0=0$ is not a point extremum.

There are functions that do not have a derivative at the extremum points.

Example. 17.9. The continuous finction $y = |x|$ at the point $x_0 = 0$, does not have a derivative, but the point $x_0 = 0$ is the minimum point of the function(fig. 17.4).

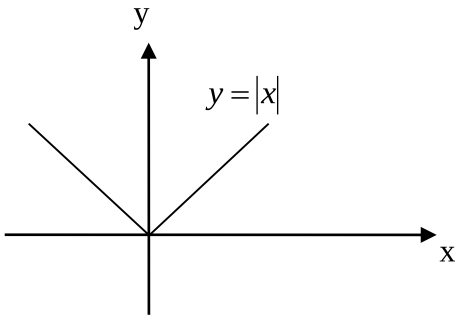


Fig.17.4.

Thus, a continuous function can have an extremum only at points where the derivative of the function is equal to zero or does not exist. Such points are called critical.

Theorem 17.9. (sufficient extremum conditions). If a continuous function $y = f(x)$ is differentiable in some δ - neighborhood of the critical point x_0 and when passing through it(from left to right) the derivative changes sign from plus to minus, then x_0 is a maximum point; from minus to plus – minimum point.

Consider the δ - neighborhood of the point x_0. Let the condition $f'(x) > 0 \ \forall x \in (x_0 - \delta, x_0)$ and $f'(x) < 0 \ \forall x \in (x_0, x_0 + \delta)$. Then the function $f(x)$ increases on the interval $(x_0 - \delta, x_0)$ and on the interval $(x_0, x_0 + \delta)$ it decreases. It follows, that the value $f(x)$ at the point x_0 is the largest on the interval $(x_0 - \delta, x_0 + \delta)$, that is all $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \ \ f(x) < f(x_0)$. This means that the point x_0 is the maximum point of the function.

A graphical interpretation of the proof of the theorem is shown in fig.17.5

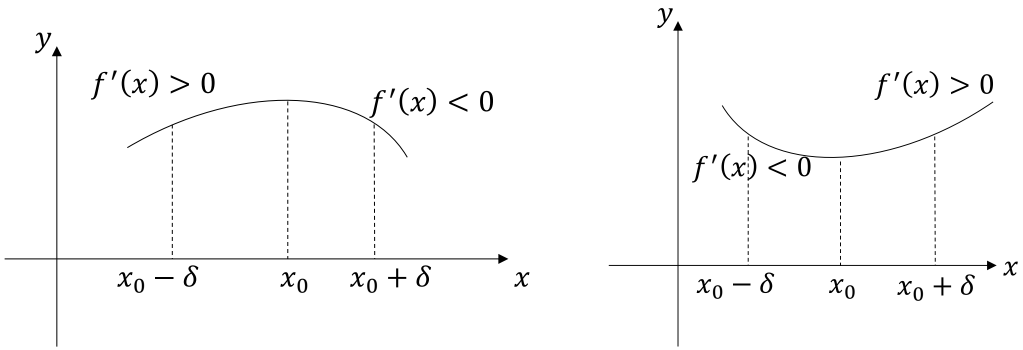


Fig. 17.5.

Similarly, the theorem is proved for the case when $f'(x) < 0 \ \ \forall x \in (x_0 - \delta, x_0)$ and $f'(x) > 0 \ \ \forall x \in (x_0, x_0 + \delta)$. To investigate a function for an extremum means to find all its extremes.

Theorems 17.8 and 17.9 imply the following rule for examinig a function for an extremum:

1) find the critical points of the function;
2) choose from them those that are interval points of the domain of definition of the function;
3) inverstigate the sign of the derivative $f'(x)$ to the left and right of each of the shosen critical points;
4) in accordance with Theorem 17.9(sufficient conditions for an extremum), write down the extremum points (if they are) and calculate the values of the function in them.

Example. 17.10. Find the extremum of a finction $y = \frac{x}{3} - x^{\frac{2}{3}}$.

Solution. It is easy to check that $D(y) = R$. We find $y' = \frac{1}{3} - \frac{2}{3\sqrt[3]{x}} = \frac{1}{3} \cdot \frac{\sqrt[3]{x} - 2}{\sqrt[3]{x}}$.

The derivative does not exist for $x_1 = 0$, and it is equal to zero for $x_2 = 8$. These points divide the entire domain of this finction into 3 intervals. We note in the figure the signs of the derivative of the function on each of the intervals of the partition. Therefore, $x_1 = 0$ is the maximum point, $y_{\max} = y(0) = 0$ and $x_2 = 8$ is the minimum point. $y_{\min} = y(8) = \frac{1}{3}$.

Sometimes it is convenient to use another sufficient criterion for the existence of an extremum, based on determining the sign of the second drrivative.

Theorem 17.10. If at the point x_0 the first derivation of the function $f(x)$ is equal to zero ($f'(x_0)=0$), and the second derivative at the point x_0 exists and is non-zero ($f'(x_0)\neq 0$), then $f''(x_0)<0$ at the point x_0, the function has a maximum, and for $f''(x_0)>0$, the function has a minimum at the point x_0.

Let, for definiteness, $f''(x_0)>0$. Since $f''(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{f'(x_0+\Delta x)-f'(x_0)}{\Delta x}>0$ then

$\dfrac{f'(x_0+\Delta x)-f'(x_0)}{\Delta x}>0$ is in a sufficiently small neighborhood of the point x_0.

If $\Delta x<0$, then $f'(x_0+\Delta x)<0$, if $\Delta x>0$, then $f'(x_0+\Delta x)>0$.

Thus, when passing through the point x_0, the first derivative changes sin from minus to plus. Therefore, by Theorem 17.10 x_0 is a minimum point.

Similary, it is proved that if $f''(x_0)<0$, then at the point x_0 the function $f(x)$ has a maximum.

17.5. The largest and smallest value of a function on a segment.

Let the function $y=f(x)$ be continuous on the segment [*a*, *b*]. As you know, such a function reaches its largest and smallest values on the segment. The function can take these values either at the interior point x_0 of the segment [*a*, *b*], or on the boundary of the segment, that is, either at $x_0=a$ or either at $x_0=b$. If at $x_0\in(a,b)$, then the point x_0 should be sought among the critical points of this function (fig.17.6).

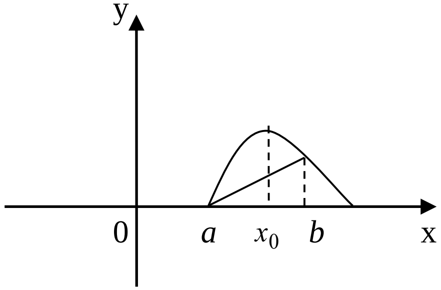


Fig. 17.6.

We get the following rule for finding the largest and smallest values of the function $y = f(x)$ on the segment [*a*, *b*]:

1. find the critical points of the function on the interval (*a*, *b*)
2. calculate the values of the function at the found critical points and at the ends of the segment $x = a,\ x = b$
3. among the values obtained, select the smallest and largest. These vill be the smallest and largest values of the function on the segment [*a*, *b*].

Example. 17.11. Find the smallest and largest value of a function $f(x)$ on a segment [*a*, *b*] $f(x) = 3x^4 + 4x^3 + 1,\ \ x \in [-2, 1]$.

1. $f'(x) = 12x^3 + 12x^2 = 12x^2(x+1)$

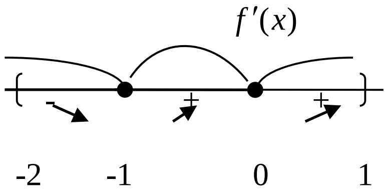


Fig. 17.7. $x_0 = -1$ - critical point.

$x_0 = -1;\ f(-1) = 0;\ a = 2;\ f(-2) = 17 : b = 1;\ f(1) = 8 \Rightarrow f_{\min} = f(-1) = 0;\ f_{\max} = f(-2) = 17$.

17.6. Convexity of the function graph. Inflection point.

The graph of a differentiable function $y = f(x)$ is called downward convex on the interval (*a*, *b*), if it is located above any its tangents on the interval (*a*, *b*).

The graph of a differentiable function $y = f(x)$ is called upward convex if it is located below any of its tangents on the interval (*a*, *b*).

The point of the graph of a continuous function $y = f(x)$, separating its parts of different convexity, is called the inflaction point.

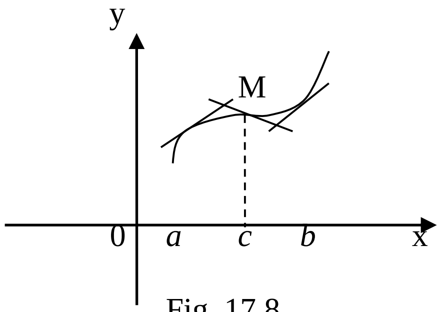


Fig. 17.8.

In fig.17.8, the curve $y = f(x)$ is called convex upward in the interval (*a*, *c*), convex downvard in the interval (*c*, *b*). The point $M(c,\ f(c))$ is the inflection point.

Theorem 17.11. If the function $y = f(x)$ at all points of the interval (*a*, *b*) has negative second derivative, $f''(x) < 0$, then the graph of the function on this interval is convex upwards.

If $f''(x) > 0\ \forall x \in (a,\ b)$, then the graph of the function on this interval is convex downvard.

Proof. Let $f''(x) < 0\ \forall x \in (a,\ b)$. Let´s take an arbitrary point M with abscissa $x_0 \in (a,\ b)$ on the graph of the function and draw a tangent through it.

Let´s show that the graph of the function is located below this tangent. To do this, compare at the point $x \in (a,\ b)$ the ordinate *y* of the curve $y = f(x)$ with the ordinate $\overline{y}$ - ordinate of the tangent at this point.

The tangent equation, as is known, has the form $\overline{y} - f(x_0) = f'(x_0)(x - x_0) \Leftrightarrow$

$\overline{y} = f(x_0) + f'(x_0)(x - x_0) \Rightarrow y - \overline{y} = f(x) - f(x_0) - f'(x_0)(x - x_0)$.

According to the Lagrange theorem $f(x) - f(x_0) = f'(c)(x - x_0)$, *c* lies between x_0 and *x*.

$\Rightarrow y-\overline{y}=f'(c)(x-x_0)-f'(x_0)(x-x_0)=(f'(c)-f'(x_0))(x-x_0)$.

According to the Lagrange theorem $f'(x)-f'(x_0)=f''(c_1)(c-x_0)$, c_1 lies between c and x_0.

Thus, $y-\overline{y}=f''(c_1)(c-x_0)(x-x_0)$.

We investigate this equality:

1. if $x>x_0 \Rightarrow x-x_0>0\ c-x_0>0$ and $f''(c_1)<0 \Rightarrow y-\overline{y}<0 \Leftrightarrow y<\overline{y}$

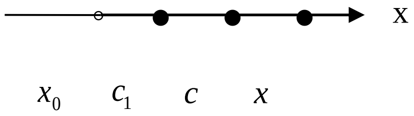


Fig. 17.9.

2. if $x<x_0 \Rightarrow x-x_0<0\ c-x_0<0 \ \Rightarrow f''(c_1)<0 \Rightarrow y-\overline{y}<0 \Leftrightarrow y<\overline{y}$

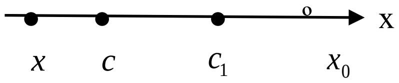


Fig. 17.10.

Thus, it is proved that at all points of the interval, the ordinate of the tangent ordinate of the graph, that is, the graph of the function is convex upwards.

Similarly, it is proved that for $f''(x) > 0$ the graph of the function is convex downwards.

To find the inflection points of the function graph, the following theorem is used.

Theorem 17.12. (a sufficient condition for existence of inflection points). If the second derivative $f''(x)$ changes sing when passing through the point x_0, at which it is equal to zero or does not exist, then the point of the graph $y = f(x)$ with abscissa x_0 is an inflection point.

Proof. Let $f''(x) < 0$ for $x < x_0$ and $f''(x) > 0$ for $x > x_0$. This means that to the left of $x = x_0$ the graph is convex upwards, and to the right it is convex downwards. Therefore, the point $(x_0,\ f(x_0))$ of the function graph is an inflection point. Similarly, it is proved that if $f''(x) > 0$ for $x < x_0$ and $f''(x) < 0$ for $x > x_0$, then the point $(x_0,\ f(x_0))$ is the inflection point of the graph of the function.

Example. 17.12. Examin the function graph for convexity an inflection point. $y = x^5 - x + 5$.

Solution. $y' = 5x^4 - 1,\ y'' = 20x^3$. $y'' = 0$ for $x = 0$.

Note, that $y'' > 0$ for $x > 0$ and $y'' < 0$ for $x < 0$

$y'' = 0$ for $x = 0$.

Therefore, the graph of the function $y = x^5 - x + 5$ in the interval $(-\infty,\ 0)$ is convex upwards, in the interval $(0,\ \infty)$ is convex downwards. Point (0, 5) is the inflection point.

17.7. Asymptotes of the graph of a function.

The construction of a graph is much easier if you know its asymptotes.

Recal that the asymptote of a curve is a straight line, the distance to which from a point lying on the curve tends to zero wih anlimited distance from the origin coordinates of this point along the curve.

Asymptotes can be vertical, oblique and horizontal.

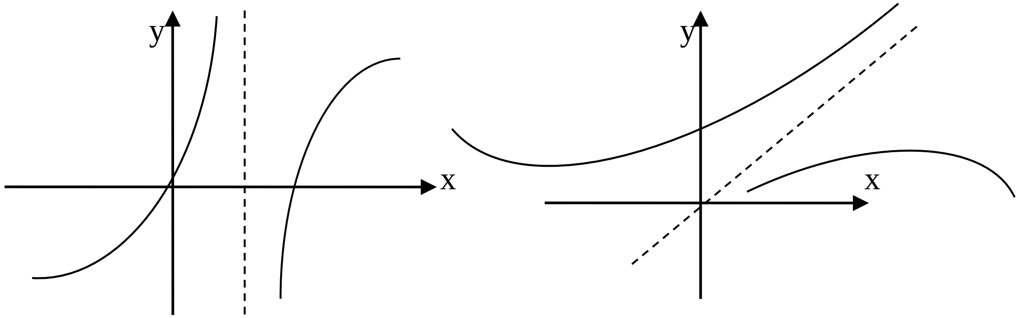


Fig. 17.11.

To find vertical asymtotes, you need to find those values of *x*, near which the function $f(x)$ increases indefinitely in absolute value. Usually these are discontinuity points of the second kind.

Example. 17.13. The curve $y=\frac{2}{x-1}$ has a vertical asymptote $x=1$ since $\lim\limits_{x\to 1+0}\frac{2}{x-1}=$

$$=\lim_{x\to +1}\frac{2}{1+0-1}=\frac{2}{0}=\infty.$$

$$\lim_{x\to 1-0}\frac{2}{x-1}=\frac{2}{1-0-1}=\frac{2}{-0}=-\infty.$$

The oblique asymptote equation is sought in the form $y = kx + b$, where $k = \lim\limits_{x \to \infty} \frac{f(x)}{x}$,

$b = \lim\limits_{x \to \infty}(f(x) - kx)$. If $k = 0$, then oblique asymptote turns into a horizontal asymptote $y = b$.

Remark. The asymptotes of graph of the function $y = f(x)$ for $x \to \infty$ and $x \to -\infty$ can be different. Therefore, when fiding *k* and *b* should be considered separately the caseses than $x \to \infty$ and $x \to -\infty$.

Example. 17.14. Find the asymtotes of the graph of a function $y = xe^x$.

$$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{xe^x}{x} = \lim_{x \to \infty} e^x = \infty.$$

Therefore, in this case there is no asymptote.

But $k = \lim\limits_{x \to -\infty} \frac{f(x)}{x} = \lim\limits_{x \to -\infty} \frac{xe^x}{x} = \lim\limits_{x \to -\infty} e^x = 0$.

$b = \lim\limits_{x \to -\infty} (e^x x - 0 \cdot x) = \lim\limits_{x \to -\infty} e^x x = \lim\limits_{x \to -\infty} \frac{x}{e^{-x}} = 0$. Therefore, the graph has a horizontal asymptote $y = 0$.

17.8. The general scheme for studying the graph and construction the graph for a function.

The following checklist is intended as a guide to sketching a curve by hand.

1. Find the domain of the function (function scope).
2. Find the points of intersection of the graph of the function with the coordinate axes, if it possible.
3. Find the intervals where $f(x) > 0$ or $f(x) < 0$.
4. Find out if the function is even, odd or general form.
5. Find the asymptotes of the graph of the function.
6. Find the monotony intervals of the function.
7. Find function extremes.

8. Find the intervals of convexity, concavity of the function and inflaction points of the graph of the function.

And based on the analysis, sketch a graph of the function.

Example. 17.15. Investigate the properties for the next function and plot its graph: $y = \frac{x}{1-x^2}$.

Solution. Investigate the function and build its graph. Let´s complete 8 points of the study.

1. The function is not defined of $x = \pm 1$. The domain of definition consists of 3 intervals and the graph of the function consists of 3 branches.
2. If $x = 0$, then $y = 0$. The graph intersects the y-axis at point O(0, 0). If $y = 0$, then $x = 0$. The graph intersects the x-axis at point O(0, 0).
3. The function is sing-positive $y > 0$ on the interval $x \in (-\infty, -1) \cup (0, 1)$. And the function is sing-negative on the interval $x \in (-1, 0) \cup (1, +\infty)$.
4. The function is odd because

$y(-x) = \frac{-x}{1-(-x)^2} = \frac{-x}{1-x^2} = -\frac{x}{1-x^2} = -y(x)$. Therefore, its graph is symmetrical about the origin O(0, 0). To build a graph, it is enough to examine it for $x \geq 0$.

5. The straight lines $x = 1$ and $x = -1$ are its vertical asymptotes. Let us find out the presence of an oblique asymptote;

$$k = \lim_{x\to\pm\infty} \frac{\frac{x}{1-x^2}}{x} = \lim_{x\to\pm\infty} \frac{1}{1-x^2} = 0.$$

$b = \lim_{x\to\pm\infty} \frac{x}{1-x^2} = 0$. Therefore, there is a horizontal asymptote and its equation has the form $y = 0$.

6. We find the intervals of increase and decrease of the function. As $y' = \frac{x^2+1}{(1-x^2)^2}$, then $y' > 0$ in the domain, than the function y is inereasing on each of the domain intervals.
7. Examine the function for an extremum. Since $y' = \frac{x^2+1}{(1-x^2)^2}$, then the critical points are $x = 1$ and $x = -1$ (y' does not exist). But they do not belong to the domain of the function. Therefore, the function has not exstremums.

8. We study the function for convexity. We find y'' :

$$y'' = \left(\frac{x^2+1}{(1-x^2)^2} \right)' = \frac{2x(1-x^2)^2 - (x^2+1)\cdot 2\cdot(1-x^2)(-2x)}{(1-x^2)^4} = \frac{2x(x^2+3)}{(1-x^2)^3} .$$

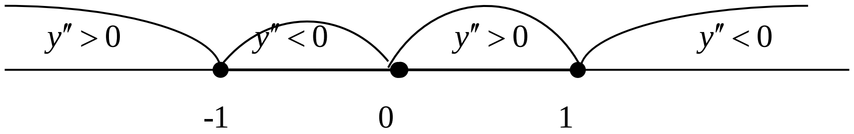


Fig. 17.12.

The second derivative is equal to zero or does not exist at points $x_1 = 0,\ x_2 = -1,\ x_3 = 1$.

Figure 17.12 shows a diagram of the change in signs of the second derivative of the function under study. Point O(0, 0) – is inflection point of the graph of the function. The graph is convex upwards at intervals $(-1,\ 0)$ and $(1,\ \infty)$ and convex downward at intervals $(-\infty, -1) \cup (0, 1)$.

The graph of the function is shown in fig. 17.13.

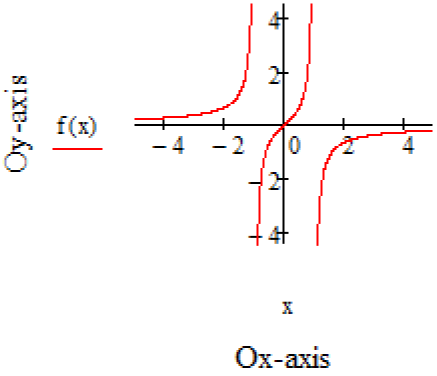


Fig. 17.13. $y = \dfrac{x}{1-x^2}$.

Практический раздел

PRACTICUM IN MATHEMATICS. PART I

Lesson 1. The Cartesian and the polar coordinate systems. Graphing functions

Classroom assignments

1.1 Scetch the graph for the functions given below:

1) $y = 2^{\log_2 \cos x}$; **2)** $y = \dfrac{x^3 - x^2}{2|x-1|}$; **3)** $y = \begin{cases} 2^{x-1}, 0 < x \le 2, \\ -x^2 - 2x, -3 < x \le 0. \end{cases}$; **4)** $y = 2x - |x-2| + 1$;

5) $y = \sqrt{\dfrac{1-\cos 2x}{2}}$; **6)** $y = \sin|x| - 1$; **7)** $y = \log_{1/2} x^2 + 1$; **8)** $y = \dfrac{1}{|x|+1}$.

1.2 Scetch the graphs of the functions specified parametrically:

1) $x = -1 + 2t,\ y = 2 - t$; **2)** $x = t,\ y = t^2 - 4$; **3)** $x = 2\cos t,\ y = \sin t$; **4)** $x = 1 - t^2,\ y = t - t^3$;

5) $x = at^2,\ y = bt^3$; **6)** $x = 2\cos^3 t,\ y = 2\sin^3 t$; **7)** $x = -1 + 2\cos t,\ y = 3 + 2\sin t$;

8) $x = 2(t - \sin t),\ y = 2(1 - \cos t)$.

1.3 Write the equations of the next curves in polar coordinates:

1) $y = x$; **2)** $y = 1$; **3)** $x^2 + y^2 = 4$; **4)** $x^2 + y^2 = 2y$; **5)** $x + y - 1 = 0$; **6)** $x^2 - y^2 = a^2$.

1.4 Scetch the graphs of the functions specified by an equation in a polar coordinate system:

1) $r = 1$; **2)** $r = 2\varphi$; **3)** $r\cos\varphi = 2$; **4)** $r = e^{\varphi}$; **5)** $r = 4\cos\varphi$; **6)** $r = 3\sin 2\varphi$;

7) $r = 2(1 + \cos\varphi)$; **8)** $r = \dfrac{6}{3 + 2\cos\varphi}$; **9)** $r = \dfrac{2}{1 + \sin\varphi}$; **10)** $r = 2\cos 3\varphi$; **11)** $r^2 = 36\sin 2\varphi$.

Homework

1.5 Scetch the graphs for the next functions:

1) $y = |x^2 - x - 2|$; **2)** $y = x + |x+3|$; **3)** $x = t^2 + 1,\ y = t$; **4)** $x = t^3,\ y = t^2$;

5) $r = 2\sin\varphi$; **6)** $r = 3(1 - \sin\varphi)$; **7)** $r = 4\cos 2\varphi$; **8)** $r = \dfrac{3}{1-\cos\varphi}$.

Answers:

1.3 1) $\varphi = \dfrac{\pi}{4}$; **2)** $r = \dfrac{1}{\sin\varphi}$; **3)** $r = 2$; **4)** $r = 2\sin\varphi$; **5)** $r = \dfrac{1}{\sin\varphi + \cos\varphi}$; **6)** $\rho^2 = \dfrac{a^2}{\cos 2\varphi}$.

Lesson 2. The matrices and the operations on them

Classroom assignments

2.1 To find $2A + 3B - C$, if

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & -3 \\ -4 & 3 & 5 \end{pmatrix},\ B = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -3 & 4 \\ 1 & -5 & 6 \end{pmatrix},\ C = \begin{pmatrix} 3 & 4 & 5 \\ 1 & -3 & 2 \\ 8 & -6 & 7 \end{pmatrix}.$$

2.2 Calculate $3A+2E$, if $A = \begin{pmatrix} 2 & 5 & -4 \\ -1 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$, where E – denotes the unit third-order matrix.

2.3 Calculate the matrix X, if

$$2 \cdot \begin{pmatrix} -1 & 3 \\ 2 & 4 \\ 0 & 5 \end{pmatrix} + \frac{1}{3} X = \begin{pmatrix} 1 & -7 \\ 2 & 8 \\ -3 & 9 \end{pmatrix}.$$

2.4 Calculate a matrix transposed to a matrix A:

1) $A=\begin{pmatrix}2&5&4\\1&3&-1\\-2&0&1\end{pmatrix}$; **2)** $A=\begin{pmatrix}1\\2\\3\end{pmatrix}$; **3)** $A=\begin{pmatrix}a&a&a\end{pmatrix}$.

2.5 For the given matrices $A=\begin{pmatrix}1&0\\-1&3\\5&0\end{pmatrix}$, $B=\begin{pmatrix}1&-1\\3&1\\7&0\end{pmatrix}$, $C=\begin{pmatrix}1&3\\5&6\\-2&3\end{pmatrix}$ you need to find:

1) $2A$; **2)** $2A+3B-C$; **3)** $-2C^T$.

2.6 For the given matrices A and B you need to find *AB* and *BA*, if:

1) $A=\begin{pmatrix}1&0&2\\0&-1&3\\4&0&5\end{pmatrix}$, $B=\begin{pmatrix}2&7&1\\3&2&-4\\1&-3&5\end{pmatrix}$; **2)** $A=\begin{pmatrix}1&1&0\\3&-1&5\end{pmatrix}$, $B=\begin{pmatrix}0&7\\3&4\\1&0\end{pmatrix}$; **3)** $A=\begin{pmatrix}3\\4\\2\end{pmatrix}$, $B=\begin{pmatrix}5&-2&3\end{pmatrix}$.

2.7 Calculate $\begin{pmatrix}3&0&1\\2&-1&0\\3&0&1\end{pmatrix}\cdot\begin{pmatrix}-1&1\\2&-2\\5&0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix}$.

2.8 Calculate those from the products of matrices AB, BA, A^2, B^2, which exist:

1) $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$; $B=\begin{pmatrix}0&-1\\1&2\end{pmatrix}$; **2)** $A=\begin{pmatrix}1&-2&3&0\end{pmatrix}$; $B=\begin{pmatrix}5\\-3\\-4\\1\end{pmatrix}$;

3) $A=\begin{pmatrix}2&0&3\\-1&2&1\end{pmatrix}$; $B=\begin{pmatrix}-4\\-3\\5\end{pmatrix}$; **4)** $A=\begin{pmatrix}3&5&-1\\2&-2&0\end{pmatrix}$; $B=\begin{pmatrix}2&4\\-3&0\\5&1\end{pmatrix}$.

2.9 Calculate the product of the matrices (*AB*)*C* and *A*(*BC*):

$$A=\begin{pmatrix}-5 & 0 & 3\\ 4 & 1 & -1\\ 2 & -3 & 2\\ 1 & 5 & 3\end{pmatrix},\ B=\begin{pmatrix}3 & 0\\ -2 & 1\\ 4 & 3\end{pmatrix},\ C=\begin{pmatrix}-2\\ 3\end{pmatrix}.$$

2.10 Show that the matrix $A=\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}$ is the root of the polynomial $f(x)=x^2-3x+5$.

2.11 Calculate the value of a matrix polynomial $f(A)$, if:

1) $f(x)=2x^2-3x+1,\ A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$;

2) $f(x)=x^2-3x+2,\ A=\begin{pmatrix}1 & -3 & 0\\ 0 & 2 & 1\\ 3 & -3 & 2\end{pmatrix}$;

3) $f(x)=2x^3-x^2+3,\ A=\begin{pmatrix}-1 & 2\\ -3 & 1\end{pmatrix}$.

Homework

2.12 Calculate: **1)** $3A-2B$, if $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$; $B=\begin{pmatrix}0 & 1\\ 1 & -2\end{pmatrix}$; **2)** $2B-5A$, if $A=\begin{pmatrix}0 & 2 & 4\\ -6 & 4 & 0\end{pmatrix}$; $B=\begin{pmatrix}0 & 5 & 10\\ -15 & 10 & 0\end{pmatrix}$.

2.13 Calculate $(A+3B)^2$, if

$$A=\begin{pmatrix}1 & 4 & 7\\ 2 & 5 & -8\\ -3 & 6 & 9\end{pmatrix},\ B=\begin{pmatrix}-2 & 1 & -1\\ 1 & 0 & 2\\ 4 & -1 & 0\end{pmatrix}.$$

2.14 Calculate those of the matrix products AB, BA, AC, CA, BC, CB that make sense, if

$$A=\begin{pmatrix}1 & -1 & 3\\ 2 & 0 & 2\end{pmatrix},\ B=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix},\ C=\begin{pmatrix}0 & 1 & 2 & 0\\ -1 & 2 & 0 & 0\\ 1 & 2 & 1 & 0\end{pmatrix}.$$

2.15 Check whether the matrices A and B are commute:

1) $A=\begin{pmatrix}1 & 2 & 3\end{pmatrix},\ B=\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$; **2)** $A=\begin{pmatrix}1 & 2\\ 3 & 5\end{pmatrix};\ B=\begin{pmatrix}-5 & 3\\ 2 & -1\end{pmatrix}$; **3)** $A=\begin{pmatrix}2 & 7 & 3\\ 3 & 9 & 4\\ 1 & 5 & 3\end{pmatrix},\ B=\begin{pmatrix}7 & -6 & 1\\ -5 & 3 & 1\\ 6 & -3 & -3\end{pmatrix}$.

2.16 Calculate the value of a matrix polynomial $f(A)$, if:

1) $f(x)=2x^2-2x+7,\ A=\begin{pmatrix}1 & 0\\ 2 & -1\end{pmatrix}$; **2)** $f(x)=3x^2+5x-2,\ A=\begin{pmatrix}2 & 3 & -3\\ 0 & 1 & 4\\ 5 & -2 & 1\end{pmatrix}$.

2.17 Calculate the matrix A^T, if:

1) $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$; **2)** $A=\begin{pmatrix}1 & -2 & 0\\ 3 & 5 & -7\\ -4 & 1 & 2\end{pmatrix}$; **3)** $A=\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}$.

Answers: **2.1** $\begin{pmatrix}-4 & -1 & -9\\ 9 & -4 & 4\\ -13 & -3 & 21\end{pmatrix}$; **2.2** $\begin{pmatrix}8 & 15 & -12\\ -3 & -7 & 3\\ 6 & 3 & -4\end{pmatrix}$; **2.3** $\begin{pmatrix}9 & -39\\ -6 & 0\\ -9 & -3\end{pmatrix}$**;** **2.4** **1)** $A^T=\begin{pmatrix}2 & 1 & -2\\ 5 & 3 & 0\\ 4 & -1 & 1\end{pmatrix}$;

2) $A^T=\begin{pmatrix}1 & 2 & 3\end{pmatrix}$; **3)** $A^T=\begin{pmatrix}a\\ a\\ a\end{pmatrix}$; **2.5 1)** $\begin{pmatrix}2 & 0\\ -2 & 6\\ 10 & 0\end{pmatrix}$; **2)** $\begin{pmatrix}4 & -6\\ 2 & 3\\ 33 & -3\end{pmatrix}$;**3)** $\begin{pmatrix}-2 & -10 & 4\\ -6 & -12 & -6\end{pmatrix}$;

2.6 1) $AB=\begin{pmatrix}4 & 1 & 11\\ 0 & -11 & 19\\ 13 & 13 & 29\end{pmatrix}$, $BA=\begin{pmatrix}6 & -7 & 30\\ -13 & -2 & -8\\ 21 & 3 & 18\end{pmatrix}$; **2)** $AB=\begin{pmatrix}3 & 11\\ 2 & 17\end{pmatrix}$, $BA=\begin{pmatrix}21 & -7 & 35\\ 15 & -1 & 20\\ 1 & 1 & 0\end{pmatrix}$;

3) $AB=\begin{pmatrix}15 & -6 & 9\\ 20 & -8 & 12\\ 10 & -4 & 6\end{pmatrix}$, $BA=(13)$; **2.7** $\begin{pmatrix}-1\\ -8\\ -1\end{pmatrix}$ **;**

2.8 1) $AB=\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$; $BA=\begin{pmatrix}-3 & -4\\ 7 & 10\end{pmatrix}$; $A^2=\begin{pmatrix}7 & 10\\ 15 & 22\end{pmatrix}$; $B^2=\begin{pmatrix}-1 & -2\\ 2 & 3\end{pmatrix}$;

2) $AB=(-1)$; $BA=\begin{pmatrix}5 & -10 & 15 & 0\\ -3 & 6 & -9 & 0\\ -4 & 8 & -12 & 0\\ 1 & -2 & 3 & 0\end{pmatrix}$; A^2 и B^2 – do not exist ;

3) $AB=\begin{pmatrix}7\\ 3\end{pmatrix}$; BA, A^2, B^2 – do not exist ;

4) $AB=\begin{pmatrix}-14 & 11\\ 10 & 8\end{pmatrix}$; $BA=\begin{pmatrix}14 & 2 & -2\\ -9 & -15 & 3\\ 17 & 23 & -5\end{pmatrix}$; A^2 и B^2 – do not exist ;

2.9 $(AB)C=A(BC)=\begin{pmatrix}33\\ -18\\ -31\\ 32\end{pmatrix}$; **2.11 1)** $\begin{pmatrix}0 & 0\\ 0 & 6\end{pmatrix}$; **2)** $\begin{pmatrix}0 & 0 & -3\\ 3 & -3 & 1\\ 0 & -12 & -3\end{pmatrix}$; **3)** $\begin{pmatrix}18 & -20\\ 30 & -2\end{pmatrix}$;

2.12 1) $\begin{pmatrix}3 & 4\\ 7 & 16\end{pmatrix}$; **2)** $\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$; **2.13** $\begin{pmatrix}96 & 12 & 2\\ -18 & 54 & -8\\ 51 & 105 & 111\end{pmatrix}$; **2.14** $BA=\begin{pmatrix}-2 & 0 & -2\\ 3 & -1 & 5\end{pmatrix}$; $AC=\begin{pmatrix}4 & 5 & 5 & 0\\ 2 & 6 & 6 & 0\end{pmatrix}$**; 2.15 1)** do not commute; **2)** do not commute;

$AB=\begin{pmatrix}-1 & 1\\ -5 & 4\end{pmatrix}\neq BA=\begin{pmatrix}4 & 5\\ -1 & -1\end{pmatrix}$; **3)** commute: $AB=BA=\begin{pmatrix}-3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}$;

2.16 1) $\begin{pmatrix}7 & 0\\ -4 & 11\end{pmatrix}$; **2)** $\begin{pmatrix}-25 & 60 & -6\\ 60 & -18 & 44\\ 70 & 23 & -63\end{pmatrix}$; **2.17 1)** $\mathrm{A}^{\mathrm{T}}=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$; **2)** $\mathrm{A}^{\mathrm{T}}=\begin{pmatrix}1 & 3 & -5\\ -2 & 5 & 1\\ 0 & -7 & 2\end{pmatrix}$;

3) $\mathrm{A}^{\mathrm{T}}=\begin{pmatrix}1\\ 2\\ 3\\ 4\end{pmatrix}$.

Lesson 3. Calculating Determinants

Classroom assignments

Evaluate second-order determinants: **1)** $\begin{vmatrix}2 & -5\\ 3 & -1\end{vmatrix}$; **2)** $\begin{vmatrix}a & 1\\ a^2 & -a\end{vmatrix}$; **3)** $\begin{vmatrix}\cos x & \sin x\\ -\cos x & \sin x\end{vmatrix}$; **4)** $\begin{vmatrix}\sqrt[4]{a} & a\\ -1 & \sqrt[4]{a^3}\end{vmatrix}$; **5)** $\begin{vmatrix}\ln x & \ln y\\ 2 & 5\end{vmatrix}$.

Evaluate by various technique: **1)** $\begin{vmatrix}-1 & 5 & 2\\ 3 & -2 & 7\\ 5 & -6 & 3\end{vmatrix}$; **2)** $\begin{vmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{vmatrix}$;

3) $\begin{vmatrix}\cos\alpha & 0 & \sin\alpha\\ 1 & 1 & 0\\ 0 & \cos\alpha & \sin\alpha\end{vmatrix}$; **4)** $\begin{vmatrix}1 & 0 & 4\\ 3 & 8 & -1\\ -1 & 4 & 2\end{vmatrix}$; **5)** $\begin{vmatrix}0 & -4 & 1\\ 1 & 3 & 1\\ 2 & 4 & 1\end{vmatrix}$.

Evaluate the determinants according to the Sarrus rule and decompose it by the elements of the 1-st line: **1)** $\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}$; **2)** $\begin{vmatrix}3 & 4 & -5\\ 8 & 7 & -2\\ 2 & -1 & 8\end{vmatrix}$.

3.1 Solve the next equation: $\begin{vmatrix} x & x+1 \\ -4 & x+1 \end{vmatrix} = 0$.

3.2 Solve the equation $\begin{vmatrix} 1 & 3 & x \\ 4 & 5 & -1 \\ 2 & -1 & 5 \end{vmatrix} = 0$.

3.3 Plot the graph for the next function $y = \begin{vmatrix} x^2 & x & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$.

3.4 Evaluate the determinants:

1) $\begin{vmatrix} 2 & 5 & 0 & 4 \\ 1 & 7 & 0 & 2 \\ 3 & 8 & 1 & 6 \\ 4 & 9 & 3 & 8 \end{vmatrix}$; **2)** $\begin{vmatrix} 2 & 4 & -1 & 2 \\ -1 & 2 & 3 & 1 \\ 2 & 5 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{vmatrix}$.

3.5 Evaluate the determinants by bringing them to a triangular form:

1) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & -1 & 7 & 4 \\ 1 & -2 & 5 & 9 \end{vmatrix}$; **2)** $\begin{vmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{vmatrix}$.

3.6 Evaluate the determinants, having previously simplified them:

1) $\begin{vmatrix} -3 & 2 & 1 & 0 \\ 2 & -2 & 1 & 4 \\ 4 & 0 & -1 & 2 \\ 3 & 1 & -1 & 4 \end{vmatrix}$; **2)** $\begin{vmatrix} 1 & 2 & 3 & 1 & 5 \\ 0 & 1 & 0 & 5 & 1 \\ 2 & 1 & 2 & 3 & 2 \\ 0 & 3 & 0 & 1 & 3 \\ 3 & 2 & 1 & 3 & 4 \end{vmatrix}$; **3)** $\begin{vmatrix} 1 & 5 & -2 & 13 \\ 0 & 2 & 7 & 1 \\ 2 & 10 & -1 & 5 \\ -3 & -15 & -6 & 13 \end{vmatrix}$;

4) $\begin{vmatrix} 3 & 1 & 2 & 4 \\ 0 & 0 & -1 & 6 \\ 2 & 1 & 3 & 1 \\ 2 & -2 & 3 & 1 \end{vmatrix}$; **5)** $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix}$; **6)** $\begin{vmatrix} 2 & 3 & -1 & 4 \\ 1 & 2 & 3 & 5 \\ -1 & 2 & 0 & 1 \\ 5 & 8 & 1 & 1 \end{vmatrix}$.

Homework

3.7 Solve the equation $\begin{vmatrix} x^2 & 1 & 4 \\ x & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$.

3.8 Evaluate $\det(AB)$ and verify that $\det(AB) = \det A \cdot \det B$, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 1 \\ 4 & -3 & 2 \end{pmatrix}.$$

3.9 Evaluate the determinants:

1) $\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 2 & 3 \\ 3 & 1 & 6 & 1 \end{vmatrix}$; **2)** $\begin{vmatrix} 2 & 3 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 2 & 1 & 0 \\ 2 & 3 & 0 & -5 \end{vmatrix}$.

3.10 Evaluate the determinants a triangular form:

1) $\begin{vmatrix} 2 & 1 & 5 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & -4 \\ 1 & 1 & 5 & 1 \end{vmatrix}$; **2)** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$.

3.11 Solve the next inequality: $\begin{vmatrix} 3 & 0 & -1 \\ 1 & x+5 & 2-x \\ 3 & -1 & 2 \end{vmatrix} \le 4$.

3.12 Evaluate the next determinants:

1) $\begin{vmatrix} 0 & 5 & 2 & 0 \\ 8 & 3 & 5 & 4 \\ 7 & 2 & 4 & 1 \\ 0 & 4 & 1 & 0 \end{vmatrix}$; **2)** $\begin{vmatrix} 7 & 3 & 2 & 6 \\ 8 & -9 & 4 & 9 \\ 7 & -2 & 7 & 3 \\ 5 & -3 & 3 & 4 \end{vmatrix}$.

Answers: 3.1 1) 13; **2)** $-2a^2$; **3)** $\sin 2x$; **4)** $2a$; **5)** $\ln\frac{x^5}{y^2}$.

3.2 1) 78; **2)** 0; **3)** $\sin 2\alpha$; **4)** 100; **5)** – 6. **3.3 1)** 0; **2)** 0.
3.4 $x=-1$; $x=-4$. **3.5** $x=-3$. **3.6** The straight line $y=2x-2$. **3.7 1)** 0; **2)** 16. **3.8 1)** 20; **2)** 27.
3.9 1) 38; **2)** 168; **3)** – 192; **4)** 75; **5)** – 12; **6)** 300. **3.10** $x_1=-1, x_2=2$. **3.11** 40. **3.12 1)** 0; **2)** 48.
3.13 1) 54; **2)** 160. **3.14** $\left(-\infty;-\frac{36}{5}\right]$. **3.15 1)** 60; **2)** 150.

Lesson 4. Inverse matrix. Solving matrix equations

Classroom assignments

4.1 Evaluate A^{-1} if they are existing for the next matrices:

1) $\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$; **2)** $\begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & -5 \\ 6 & 1 & -2 \end{pmatrix}$; **3)** $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix}$. **4)** $\begin{pmatrix} -3 & 1 & 9 \\ -5 & -3 & 8 \\ -4 & -1 & 5 \end{pmatrix}$; **5)** $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

4.2 Evaluate the inverse matrix, if it exists:

1) $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$; **2)** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; **3)** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$; **4)** $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ -4 & -14 & -6 \end{pmatrix}$.

4.3 Solve the next matrix equations:

1) $\begin{pmatrix} -1 & 4 \\ 3 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$; **2)** $\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} -1 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}$;

3) $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix} \cdot X + \begin{pmatrix} -1 & 2 \\ -1 & 4 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -1 & 2 \\ 5 & 12 \end{pmatrix}$.

4.4 Solve the next matrix equations:

1) $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$; **2)** $X \cdot \begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; **3)** $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \cdot X \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$.

4.5 Solve the next matrix equations:

1) $X \cdot \begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 2 & -1 \\ -1 & -2 & 4 \end{pmatrix}$; **2)** $X \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$; **3)** $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 0 & -2 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix}$.

Homework

4.6 Evaluate matrices inverse of the matrices given below, if they are exist:

1) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$; **2)** $\begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{pmatrix}$ **3)** $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix}$; **4)** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

4.7 Solve the next matrix equations:

1) $X \cdot \begin{pmatrix} 5 & 3 & 1 \\ 1 & -3 & -2 \\ -5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 3 & 0 \\ -5 & 9 & 0 \\ -2 & 15 & 0 \end{pmatrix}$; **2)** $\begin{pmatrix} 5 & 4 \\ -1 & -2 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{pmatrix}$;

3) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \cdot X \begin{pmatrix} 2 & -2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$; **4)** $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 0 & -2 & 1 \end{pmatrix} \cdot X \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$.

Answers: **4.1** **1)** $\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$; **2)** does not exist; **3)** $-\frac{1}{38}\begin{pmatrix} -10 & 4 & -2 \\ 7 & 1 & -10 \\ -8 & -12 & 6 \end{pmatrix}$;

4) $-\frac{1}{49}\begin{pmatrix} -7 & -14 & 35 \\ -7 & 21 & -21 \\ -7 & -7 & 14 \end{pmatrix}$; **5)** $-\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$.

4.2 1) $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$; **2)** $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$; **3)** $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$; **4)** does not exist.

4.3 1) $\begin{pmatrix} -\frac{11}{15} & 1 \\ \frac{1}{15} & 0 \end{pmatrix}$; **2)** $\begin{pmatrix} -\frac{3}{4} & \frac{3}{4} \\ -\frac{1}{8} & \frac{5}{8} \end{pmatrix}$; **3)** $\begin{pmatrix} \frac{5}{13} & 3 \\ \frac{5}{13} & -1 \\ \frac{30}{13} & 4 \end{pmatrix}$.

4.4 1) $\begin{pmatrix} -3 & 3 \\ -1 & 3 \end{pmatrix}$; **2)** $\begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix}$; **3)** $\begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}$.

4.5 1) $\begin{pmatrix} 20 & -15 & 13 \\ -17 & 13 & -10 \\ -8 & 5 & -4 \end{pmatrix}$; **2)** $\begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}$; **3)** $\begin{pmatrix} 6 \\ -5 \\ -3 \end{pmatrix}$.

4.6 1) $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$; **2)** $\begin{pmatrix} 1 & -1 & 1 \\ -38 & 41 & -34 \\ 27 & -29 & 24 \end{pmatrix}$; **3)** $\begin{pmatrix} -4 & 3 & -2 \\ -8 & 6 & -5 \\ -7 & 5 & -4 \end{pmatrix}$; **4)** does not exist.

4.7 1) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$; **2)** $-\frac{1}{6}\cdot\begin{pmatrix} 10 & 4 & -2 \\ -14 & -8 & -2 \end{pmatrix}$; **3)** $\begin{pmatrix} 5/2 & 1 \\ 2 & 1 \end{pmatrix}$; **4)** $\begin{pmatrix} -\frac{1}{7} & \frac{4}{7} & 1 \\ \frac{2}{7} & -\frac{1}{7} & -1 \\ \frac{4}{7} & -\frac{2}{7} & -1 \end{pmatrix}$.

Lesson 5. Solving non-degenerate systems of linear equations

Classroom assignments

5.1 Solve the systems according to Kramer's formulas and matrix way:

1) $\begin{cases} x+2y+3z=5, \\ 4x+5y+6z=8, \\ 7x+8y \qquad =2; \end{cases}$

2) $\begin{cases} 2x_1-3x_2+x_3=-7, \\ x_1+2x_2-3x_3=14, \\ -x_1-x_2+5x_3=-18; \end{cases}$

3) $\begin{cases} x_1+2x_2+3x_3=3, \\ 2x_1+6x_2+4x_3=12, \\ 3x_1+10x_2+8x_3=21. \end{cases}$

4) $\begin{cases} x_1-2x_2+x_3=0, \\ 2x_1-x_2 \qquad =1, \\ 3x_1+2x_2-x_3=4. \end{cases}$

5) $\begin{cases} -2x+2y-z+7=0, \\ x-3y+z-6=0, \\ 3x+y+2z-7=0. \end{cases}$

6) $\begin{cases} 3x_1+x_2+x_3=2, \\ x_1-2x_2+2x_3=-1, \\ 4x_1-3x_2-x_3=5. \end{cases}$

7) $\begin{cases} 2x-y+5z=4, \\ 3x-y+5z=0, \\ 5x+2y+13z=2. \end{cases}$

8) $\begin{cases} x_1+2x_2=8, \\ 3x_1+4x_2=18. \end{cases}$

9) $\begin{cases} 2x_1-3x_2+x_3=5, \\ x_1+4x_2-x_3=-3, \\ 3x_1+2x_2+3x_3=1. \end{cases}$

10) $\begin{cases} 2x-y+2z=1, \\ 3x+2y-z=9, \\ x-4y+3z=-5. \end{cases}$

11) $\begin{cases} 7x_1-2x_2-3x_3+3=0, \\ x_1+5x_2+x_3-14=0, \\ 3x_1+4x_2+2x_3-10=0. \end{cases}$

Homework

Determine whether the given linear system has exactly one solution and solve it.

1) $\begin{cases} 4x_1+2x_2-x_3=0, \\ x_1+2x_2+x_3=1, \\ x_2-x_3=-3. \end{cases}$

2) $\begin{cases} 2x_1-x_2=5, \\ x_1+4x_3=0, \\ x_2+2x_3=-1. \end{cases}$

3) $\begin{cases} 2x+y \quad =5, \\ x+\ \ 3z=16, \\ 5y-z=10. \end{cases}$

4) $\begin{cases} x_1+x_2-2x_3=6, \\ 2x_1+3x_2-7x_3=16, \\ 5x_1+2x_2+x_3=16. \end{cases}$

Answers: 5.**1** **1)** (–2;2;1); **2)** (1;2;–3); **3)** (–3;3;0); **4)** $x_1=x_2=x_3=1$; **5)** $x=2,\ y=-1,\ z=1$**;** **6)** $x_1=1,\ x_2=0,\ x_3=-1$**;** **7)** $x=-4,\ y=-2,\ z=2$; **8)** $x_1=2,\ x_2=3$; **9)** $x_1=1,\ x_2=-1,\ x_3=0$; **10)** $x=2,\ y=1,\ z=-1$; **11)** $x_1=0,\ x_2=3,\ x_3=-1$.

5.2 1) $x_1=1,\ x_2=-1,\ x_3=2$; **2)** $x_1=\frac{8}{3},\ x_2=\frac{1}{3},\ x_3=-\frac{2}{3}.$; **3)** $x=1,\ y=3,\ z=5$;

4) $x_1=3,\ x_2=1,\ x_3=-1$.

Lesson 6. Matrix rank

Classroom assignments

6.1 Find the rank for the nrxt matrices:

$$\textbf{1)}\begin{pmatrix}3 & -1 & 2\\ 4 & -3 & 3\\ 1 & 3 & 0\end{pmatrix};\quad \textbf{2)}\begin{pmatrix}2 & -1 & 5 & 6\\ 1 & 1 & 3 & 5\\ 1 & -5 & 1 & -3\end{pmatrix}.$$

6.2 Find the ranks of matrices using elementary transformations or the method of bordering minors and specify any basic minor.

$$\textbf{1)}\begin{pmatrix}-1 & 2 & 4 & 5\\ 2 & -1 & 0 & 6\\ 2 & -4 & -8 & 4\end{pmatrix};\quad \textbf{2)}\begin{pmatrix}-8 & 1 & -7 & -5 & -5\\ -2 & 1 & -3 & -1 & -1\\ 1 & 1 & -1 & 1 & 1\end{pmatrix};\ \textbf{3)}\begin{pmatrix}1 & 3 & 5 & -1\\ 2 & -1 & -3 & 4\\ 5 & 1 & -1 & 7\\ 7 & 7 & 9 & 1\end{pmatrix};$$

$$\textbf{4)}\begin{pmatrix}3 & -1 & 3 & 2 & 5\\ 5 & -3 & 2 & 3 & 4\\ 1 & -3 & -5 & 0 & -7\\ 7 & -5 & 1 & 4 & 1\end{pmatrix};\quad \textbf{5)}\begin{pmatrix}-1 & 0 & 2 & 4\\ 2 & 1 & 3 & -1\\ 1 & 1 & 5 & 3\\ -4 & -2 & -6 & 2\\ 0 & 1 & 7 & 7\end{pmatrix}.$$

6.3 At what values of the parameter λ the rank of the next matrix equals two:

$$\textbf{1)}\begin{pmatrix}1 & 3 & -4\\ \lambda & 0 & 1\\ 4 & 3 & -3\end{pmatrix};\quad \textbf{2)}\begin{pmatrix}\lambda & 2 & 3\\ 0 & \lambda-2 & 4\\ 0 & 0 & 7\end{pmatrix}?$$

6.4 Are the next equalities true? $r_{AB} \le r_A,\ r_{AB} \le r_B$.

$$A=\begin{pmatrix}1 & 0 & 2\\ -1 & 2 & 3\\ -3 & 1 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & 2 & 4\\ 3 & -1 & 5\\ 2 & 0 & 1\end{pmatrix}.$$

Homework

6.5 Evaluate the ranks of the matrices and specify some basic minor.

1) $\begin{pmatrix}2 & -1 & 3\\ 4 & -2 & 0\\ 0 & 0 & -6\\ -4 & 2 & 1\end{pmatrix}$; **2)** $\begin{pmatrix}-2 & 1 & -1 & 3 & 1\\ 1 & 0 & 2 & -1 & 1\\ 1 & 3 & 11 & 2 & -5\\ -1 & 4 & 10 & 5 & -4\end{pmatrix}$; **3)** $\begin{pmatrix}1 & 2 & -1 & -2\\ 0 & 1 & 2 & 3\\ 1 & 2 & -1 & 0\\ 1 & 3 & 1 & 1\\ 2 & 5 & 0 & 1\end{pmatrix}$.

6.6 Test the fairness of inequality $r_{A+B} \le r_A + r_B$, if

$$A=\begin{pmatrix}1 & -1 & 1\\ 2 & -1 & 3\\ 3 & -2 & 4\end{pmatrix},\ B=\begin{pmatrix}1 & -1 & 1\\ 2 & -2 & 2\\ -1 & 1 & -1\end{pmatrix}.$$

Answers: **6.1** **1)** 2; **2)** 2. **6.2** **1)** $r=3,\ \begin{vmatrix}-1 & 2 & 5\\ 2 & -1 & 6\\ 2 & -4 & 4\end{vmatrix}$; **2)** $r=2,\ \begin{vmatrix}-8 & 1\\ -2 & 1\end{vmatrix}$; **3)** $r=3,\ \begin{vmatrix}1 & 3 & -1\\ 2 & -1 & -3\\ 7 & 7 & 1\end{vmatrix}$;

4) $r=3,\ \begin{vmatrix}3 & -1 & 5\\ 5 & -3 & 4\\ 7 & -5 & 1\end{vmatrix}$; **5)** $r=2,\ \begin{vmatrix}-1 & 0\\ 2 & 1\end{vmatrix}$. **6.3** **1)** $\lambda=3$; **2)** $\lambda=0,\ \lambda=2$. **6.5** **1)** 2; **2)** 3; **3)** 3.

Lesson 7. Solving arbitrary and homogeneous systems of linear equations

Classroom assignments

7.1 Solve the next systems:

1) $\begin{cases} 2x - y + z = -2, \\ x + 2y + 3z = -1, \\ x - 3y - 2z = 3. \end{cases}$ **2)** $\begin{cases} 2x_1 + 7x_2 + 3x_3 + x_4 = 6, \\ 3x_1 + 5x_2 + 2x_3 + 2x_4 = 4, \\ 9x_1 + 4x_2 + x_3 + 7x_4 = 2. \end{cases}$ **3)** $\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 + x_5 = 1, \\ 2x_1 - 3x_2 + 2x_3 + x_4 - x_5 = 3. \end{cases}$

4) $\begin{cases} x_1 + 2x_2 + x_3 - 3x_4 + x_5 = 1, \\ x_1 - 3x_2 + x_3 - 2x_4 + x_5 = -3, \\ x_1 + 7x_2 + x_3 - 4x_4 + x_5 = 5. \end{cases}$ **5)** $\begin{cases} 3x_1 - x_2 + x_3 + 2x_5 = 18, \\ 2x_1 - 5x_2 + x_4 + x_5 = -7, \\ x_1 - x_4 + 2x_5 = 8, \\ 2x_2 + x_3 + x_4 - x_5 = 10, \\ x_1 + x_2 - 3x_3 + x_4 = 1. \end{cases}$ **6)** $\begin{cases} x_1 - x_2 + x_3 - x_4 = -2, \\ x_1 + 2x_2 - 2x_3 - x_4 = -5, \\ 2x_1 - x_2 - 3x_3 + 2x_4 = -1, \\ x_1 + 2x_2 + 3x_3 - 6x_4 = -10. \end{cases}$

7) $\begin{cases} x_1 - 3x_2 + 4x_3 - x_4 = 2, \\ 2x_1 + 3x_2 + x_3 + 5x_4 = 3, \\ 3x_1 + \quad + 5x_3 + 4x_4 = 6. \end{cases}$ **8)** $\begin{cases} x_1 - 5x_2 + 3x_3 - x_4 = 1, \\ 2x_1 - 10x_2 + \quad 3x_4 = 0, \\ 4x_1 - 20x_2 + 6x_3 + x_4 = 2. \end{cases}$

7.2 Solve a homogeneous system and find the fundamental system of solutions.

1) $\begin{cases} x_1 + 2x_2 - x_3 = 0, \\ 2x_1 + 9x_2 - 3x_3 = 0. \end{cases}$ **2)** $\begin{cases} 3x_1 + 2x_2 + x_3 = 0, \\ 2x_1 + 5x_2 + 3x_3 = 0, \\ 3x_1 + 4x_2 + 2x_3 = 0. \end{cases}$ **3)** $\begin{cases} 2x_1 + 2x_2 - x_3 + 3x_4 = 0, \\ x_1 + x_2 + 3x_3 - x_4 = 0. \end{cases}$

4) $\begin{cases} x_1 + 4x_2 - 3x_3 + 6x_4 = 0, \\ 2x_1 + 5x_2 + x_3 - 2x_4 = 0, \\ x_1 + 7x_2 - 10x_3 + 20x_4 = 0. \end{cases}$ **5)** $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 0, \\ 2x_1 + 4x_2 - x_3 - x_4 = 0, \\ 3x_1 + 6x_2 - 4x_3 \quad = 0. \end{cases}$ **6)** $\begin{cases} 3x_1 + x_2 - 2x_3 + x_4 - x_5 = 0, \\ 6x_1 + 3x_2 + x_3 - 2x_4 + x_5 = 0, \\ x_1 + 2x_2 - x_3 + x_4 + x_5 = 0. \end{cases}$

7.3 Solve the next systems by using the Gauss method:

1) $\begin{cases} x_1 + x_2 - x_3 = -4, \\ x_1 + 2x_2 - 3x_3 = 0, \\ -2x_1 \quad - 2x_3 = 16; \end{cases}$ **2)** $\begin{cases} x_1 + x_2 + x_3 = 3, \\ 2x_1 - x_2 + x_3 = 2, \\ x_1 + 4x_2 + 2x_3 = 5; \end{cases}$ **3)** $\begin{cases} 3x - y + 2z = 0, \\ 4x - 3y + 3z = 0, \\ x + 3y \quad = 0; \end{cases}$ **4)** $\begin{cases} x + 2y + 3z = 6, \\ 4x + 5y + 6z = 9, \\ 7x + 8y \quad = -6. \end{cases}$

Homework

7.4 Solve the next systems:

1) $\begin{cases} x_1+2x_2+x_3=-1, \\ 2x_1+3x_2+5x_3=3, \\ 3x_1+5x_2+6x_3=7. \end{cases}$ **2)** $\begin{cases} x_1-x_2+3x_3=1, \\ 2x_1+3x_2-2x_3=2, \\ 4x_1+x_2+4x_3=4. \end{cases}$ **3)** $\begin{cases} 2x_1+3x_2=1, \\ 3x_1+4x_2=1, \\ x_1+2x_2=1, \\ 4x_1+5x_2=1. \end{cases}$ **4)** $\begin{cases} x_1-5x_2+3x_3-x_4=1, \\ 2x_1-10x_2+3x_4=0, \\ 4x_1-20x_2+6x_3+x_4=2. \end{cases}$

7.4 Solve the next systems:

1) $\begin{cases} x_1+2x_2-x_3=0, \\ 3x_1-x_2+2x_3=0, \\ 4x_1+x_2+3x_3=0. \end{cases}$ **2)** $\begin{cases} 3x_1-x_2+2x_3+x_4=0, \\ x_1+2x_2-4x_3-2x_4=0. \end{cases}$

Answers: 7.1 **1)** The system is incompatible; **2)** $\left\{\left(\frac{C_1-9C_2-2}{11},\ \frac{10-5C_1+C_2}{11},\ C_1,C_2\right)\middle|\forall C_1,C_2\in R\right\}$**;**

3) $\left\{\left(\frac{9-C_1-14C_2-C_3}{7},\ \frac{4C_1-7C_2-3C_3-1}{7},\ C_1,\ C_2,\ C_3\right)\middle|\ \forall C_1,\ C_2,\ C_3\in R\right\}$**;**

4) $\left\{\left(\frac{-3-5C_1+13C_2-5C_3}{5},\ \frac{4+C_2}{5},\ C_1,\ C_2,\ C_3\right)\middle|\forall C_1,C_2,C_3\in R\right\}$**;**

5) $x_1=5,\quad x_2=4,\quad x_3=3,\quad x_4=1,\quad x_5=2$; **6)** $\{(C, C+1, C+2, C+3)|\forall C\in R\}$ **;**

7) The system is incompatible; **8)** $\left\{\left(C_1,C_2,\frac{3-5C_1+25C_2}{9},\frac{10C_2-2C_1}{3}\right)\middle|\forall C_1,C_2\in R\right\}$ **.**

7.2 1) $\left\{\left(\frac{3}{5}C_1,\frac{C_1}{5},C_1\right)\middle|\forall C_1\in R\right\}$; $(3,1,5)$; **2)** $x_1=x_2=x_3=0$;

3) $\left\{\left(\frac{-7C_1-8C_2}{7}, C_1, \frac{5C_2}{7}, C_2\right) \middle| \forall C_1C_2 \in R\right\}$; $(-1, 1, 0, 0)$; $\left(-\frac{8}{7}, 0, \frac{5}{7}, 1\right)$;

4) $\left\{\left(\frac{-19C_1+38C_2}{3}, \frac{7C_1-14C_2}{2}, C_1, C_2\right) \middle| \forall C_1C_2 \in R\right\}$, $\left(-\frac{19}{3}, \frac{7}{2}, 1, 0\right)$, $\left(\frac{38}{3}, -7, 0, 1\right)$;

5) $\left\{\left(C_1, C_2, \frac{3C_1+6C_2}{4}, \frac{5C_1+10C_2}{4}\right) \middle| \forall C_1, C_2 \in R\right\}$, $\left(1,0,\frac{3}{4},\frac{5}{4}\right)$, $\left(0,1,\frac{3}{2},\frac{5}{2}\right)$**;**

6) $\left\{\left(\frac{8C_1+9C_2}{26}, -\frac{6C_1+23C_2}{26}, \frac{22C_1-11C_2}{26}, C_1, C_2\right) \middle| \forall C_1C_2 \in R\right\}$, $\left(\frac{4}{13}, -\frac{3}{13}, \frac{11}{13} 1, 0\right)$, $\left(\frac{9}{26}, -\frac{23}{26}, -\frac{11}{26}, 0, 1\right)$**.**

7.3 1) $(-C-8; 2C+4; C)$; $C \in R$; **2)** the system is incompatible; **3)** $(-3C; C; 5C)$; $C \in R$; **4)** (–2;1;2).

7.4 1) the system is incompatible; **2)** $\left\{\left(\frac{5-7C}{5}, \frac{8C}{5}, C\right) \mid \forall C \in R\right\}$; **3)** $x_1 = -1, x_2 = 1$;

4) $\left\{\left(C_1, C_2, \frac{3-5C_1+25C_2}{9}, \frac{10C_2-2C_1}{3}\right) \middle| \forall C_1, C_2 \in R\right\}$.

7.5 1) $x_1 = 0, x_2 = 0, x_3 = 0$; **2)** $\{(0, 2C_1+C_2, C_1, C_2) \mid \forall C_1, C_2 \in R\}$.

Lesson 8. Vectors. Linear operations on the vectors. Scalar product of the vectors

Classroom assignments

8.1 Determine for which vectors $\vec{a}$ and $\vec{b}$ the following conditions are met:

1) $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$; **2)** $|\vec{a}+\vec{b}| = |\vec{a}| - |\vec{b}|$; **3)** $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$; **4)** $|\vec{a}+\vec{b}| = 0$; **5)** $\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{b}}{|\vec{b}|}$.

8.2 Vectors are given $\vec{a} = 3\vec{i} - 2\vec{j} + 6\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j}$. Define projections on the coordinate axes of the following vectors:

1) $-\frac{1}{2}\vec{b}$; **2)** $2\vec{a}$; **3)** $2\vec{a}+3\vec{b}$.

8.3 Check the collinearity of vectors $\vec{a}(2;-1;3)$ and $\vec{b}(-6;3;-9)$. Establish which one is longer than the other and how many times, how they are directed - in one direction or in opposite directions.

8.4 For what α and β the vectors $\vec{a}=\alpha\vec{i}-5\vec{j}+3\vec{k}$ and $\vec{b}=\vec{i}+2\vec{j}-\alpha\vec{k}$ are orthogonal? For what α and β the vectors $\vec{a}=\alpha\vec{i}-5\vec{j}+3\vec{k}$ and $\vec{b}=\vec{i}+2\vec{j}-\alpha\vec{k}$ are collinear?

8.5 Evaluate the vector cosine guides $\vec{a}(6;-2;-3)$.

8.6 Define the modules of the sum and difference of vectors $\vec{a}=3\vec{i}-5\vec{j}+8\vec{k}$ and $\vec{b}=-\vec{i}+\vec{j}-4\vec{k}$.

8.7 The vertices of the triangle are given $A(4;-1;2)$, $B(0;1;-3)$, $C(6;5;3)$. Find:1) the coordinates of the vector $\overrightarrow{AD}$, where *AD* is the median of the triangle; 2) the coordinates for the point *O* of the intersection for the medians of this triangle.

8.8 Consider the nexr points: $A(4;4;0)$, $B(0;0;0)$, $C(0;3;4)$, $D(1;4;4)$. Prove that ABCD is an isosceles trapezoid.

8.9 Given a triangle with vertices at points $A(2;3;-1)$, $B(4;1;-2)$, $C(1;0;2)$. To find: a) inner angle at vertex *C*; b) square of a triangle *ACC*; c) length of height lowered from vertex *C* on *AB*.

8.10 Given points $A(-1;2;1), B(2;1;-3), C(3;0;5)$. Pick a point so that the quadrilateral is a parallelogram.

8.11 Evaluate the next expression: $(\vec{m}+2\vec{n},\vec{m}-\vec{n})$, if $\vec{m}=2\vec{a}+\vec{b}$, $\vec{n}=\vec{a}-3\vec{b}$, $|\vec{a}|=|\vec{b}|=2$; $(\vec{a},\wedge\vec{b})=\frac{\pi}{3}$.

8.12 The vertices of the quadrilateral are given $A(1;-2;2), B(1;4;0), C(-4;1;1)$, $D(-5;-5;3)$. Prove that its diagonals *AC* and *BD* are mutually perpendicular.

8.13 Calculate the inner corners of a triangle *ACC*, if $A(1;2;1)$, $B(3;-1;7)$, $C(7;4;-2)$. Make sure that this triangle is isosceles.

8.14 Calculate vector projection $\vec{a}=5\vec{i}+2\vec{j}-5\vec{k}$ on the vector axis $\vec{b}=2\vec{i}-\vec{j}+2\vec{k}$.

8.15 Vectors are given $\vec{a}=(1;-3;4)$, $\vec{b}=(3;-4;2)$, $\vec{c}=(-1;1;4)$. To find $pr_{\vec{b}+\vec{c}}\vec{a}$.

8.16 What kind of work does power do $\vec{F}=(2;-1;-4)$, when the point of its application, moving in a straight line, moves from the point $A=(1;-2;3)$ to the point $B=(5;-6;1)$?

Homework

8.17 Find the length of the diagonals for the parallelogram constructed on the vectors $\vec{a}(3;-5;8)$ and $\vec{b}(-1;1;-4)$. Calculate the cosine of the angle between its diagonals.

8.18 Three vectors are given $\vec{a}(-2;1;1)$, $\vec{b}(1;5;0)$ and $\vec{c}(4;4;-2)$. Calculate $pr_{\vec{c}}(3\vec{a}-2\vec{b})$.

8.19 At what value α the vectors $\vec{a}=\alpha\vec{i}-3\vec{j}+2\vec{k}$ and $\vec{b}=\vec{i}+2\vec{j}-\alpha\vec{k}$ are mutually perpendicular?

8.20 Vectors $\vec{a}$ and $\vec{b}$ form the angle $\varphi=\frac{\pi}{6}$. Knowing that $|\vec{a}|=\sqrt{3}, |\vec{b}|=1$, calculate the angle α between vectors $\vec{p}=\vec{a}+\vec{b}$ and $\vec{q}=\vec{a}-\vec{b}$.

8.21 Find the coordinates for the vector $\vec{b}$ that collinear to the vector $\vec{a}=(2;1;-1)$, provided that $(\vec{a},\vec{b})=3$.

8.22 Given points $A(-1;0;2)$, $B(2;3;-4)$, $C(2;3;4)$. Find the vector coordinates $\overrightarrow{AD}$, if it is known that the point *D* divides the segment *BC* with respect to $\lambda=3$.

8.23 Find the direction cosines of the vector $\overrightarrow{AB}$, if $A(3;4;-5)$, $B(-1;8;-3)$.

8.24 Find the vector $\vec{b}$, orthogonal to the vector $\vec{a}=\vec{i}+2\vec{j}-\vec{k}$ and satisfying the conditions $(\vec{b},\vec{i})=3$; $(\vec{b},\vec{j})=2$.

Answers: 8.1 1) $\vec{a}\uparrow\uparrow\vec{b}$; **2)** $\vec{a}\uparrow\downarrow\vec{b}$, $|\vec{a}|\geq|\vec{b}|$; **3)** $\vec{a}\perp\vec{b}$; **4)** $\vec{a}=-\vec{b}$; **5)** $\vec{a}\uparrow\uparrow\vec{b}$; $\vec{a}\neq 0$; $\vec{b}\neq 0$.

8.2 1) $\left(1;-\frac{1}{2};0\right)$; **2)** $(6;-4;12)$; **3)** $(0;-1;12)$.

8.3 The vectors oppositely directed, the vector $\vec{b}$ 3 times longer than the vector $\vec{a}$.

8.4 1) $\alpha = -5$; 2) $\alpha = -10; \beta = -\frac{3}{5}$. **8.5** $\cos\alpha = \frac{6}{7}$; $\cos\beta = -\frac{2}{7}$; $\cos\beta = -\frac{3}{7}$.

8.6 $\left|\vec{a}+\vec{b}\right| = 6$; $\left|\vec{a}-\vec{b}\right| = 14$. **8.7 1)** $\overrightarrow{AD} = (-1;4;-2)$; **2)** $O = \left(\frac{10}{3};\frac{5}{3};\frac{2}{3}\right)$. **8.8** $\vec{a} \uparrow\downarrow \vec{b}, \left|\vec{a}\right| \geq \left|\vec{b}\right|$.

8.9 a) $\arccos\frac{18}{\sqrt{494}}$; b) $\frac{\sqrt{170}}{2}$; c) $\frac{\sqrt{170}}{3}$. **8.10** $D(0;1;9)$. **8.11** – 42.

8.13 $\cos\angle A = -\frac{12}{49}$; $\cos\angle B = \frac{\sqrt{122}}{14}$; $\cos\angle C = \frac{\sqrt{122}}{14}$. **8.14** $-\frac{2}{3}$. **8.15** 5. **8.16** 20.

8.17 $|\vec{a}+\vec{b}| = 6$, $|\vec{a}-\vec{b}| = 14$, $\cos\varphi = \frac{20}{21}$. **8.18** $pr_{\vec{c}}(3\vec{a}-2\vec{b}) = -11$. **8.19** $\alpha = -6$. **8.20** $\alpha = \arccos\frac{2}{\sqrt{7}}$. **8.21** $\vec{b} = \left(1;\frac{1}{2};-\frac{1}{2}\right)$. **8.22** $\overrightarrow{AD}(3;3;0)$. **8.23** $\cos\alpha = -\frac{2}{3}$; $\cos\beta = \frac{2}{3}$; $\cos\gamma = \frac{1}{3}$. **8.24** $\vec{b}(3;2;7)$.

Lesson 9. Vector and mixed vector products

Classroom assignments

9.1 Vectors $\vec{a}$ and $\vec{b}$ are orthogonal. Knowing that $|\vec{a}| = 3, |\vec{b}| = 4$, calculate: **1)** $|[\vec{a},\vec{b}]|$; **2)** $|[\vec{a}+\vec{b}, \vec{a}-\vec{b}]|$; **3)** $|[(3\vec{a}+\vec{b}),(\vec{a}-\vec{b})]|$.

9.2 Vectors are given $\vec{a} = (3;-1;-2), \vec{b} = (1;2;-1)$. Find the coordinates of vector products: **1)** $[\vec{a},\vec{b}]$; **2)** $[2\vec{a}+\vec{b}, \vec{b}]$; **3)** $[2\vec{a}-\vec{b}, 2\vec{a}+\vec{b}]$.

9.3 Find any non-zero vector $\vec{c}$, perpendicular to the vectors $\vec{a} = (1;2;3)$ and $\vec{b} = (0;2;5)$.

9.4 Calculate the sine of the angle formed by the vectors $\vec{a} = 6\vec{j}+\vec{k}$ and $\vec{b} = \vec{i}+3\vec{j}$.

9.5 The vertices of the triangle are given $A(1;-1;2), B(5;-6;2), C(1;3;-1)$. Calculate the area of the triangle and the length of the height dropped from the vertex B to the side AC.

9.6 Prove the validity of the identity $\left[\vec{a}-\vec{b},\, \vec{a}+\vec{b}\right]=2\left[\vec{a},\vec{b}\right]$ and find out its geometric meaning.

9.7 Consider the next vectors: $\vec{a}=2\vec{i}+\vec{j}-\vec{k}; \vec{b}=\vec{i}-\vec{j}+3\vec{k}, \vec{c}=\vec{j}+\vec{k}$. Find: **1)** $\left[\left[\vec{a},\vec{b}\right],\vec{c}\right]$; **2)** $\left[\vec{a},\left[\vec{b},\vec{c}\right]\right]$.

9.8 It is known that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Prove that $\left[\vec{a},\vec{b}\right]=\left[\vec{c},\vec{a}\right]=\left[\vec{b},\vec{c}\right]$.

9.9 The force $\vec{F}=(3;4;2)$ is applied to the point $C=(-2;1;-2)$. Determine the magnitude and guide cosines of the moment of force relative to the origin.

9.10 Find out if the vectors are complanar:

a) $\vec{a}=(0;1;1),\ \vec{b}=(1;1;1), \vec{c}=(1;0;0)$; b**)** $\vec{a}=(4;-2;0),\ \vec{b}=(-3;6;3),\ \vec{c}=(1;4;-5)$.

9.11 Prove that the four points $A(1;\ 2;\ -1), B(0;\ 1;\ 5), C(-1;\ 2;\ 1),\ D(2;1;3)$ lie on the same plane.

9.12 Tetrahedron vertices: $A(2;3;1), B(4;1;-2), C(6;3;7),\ D(-5;-4;8)$. To find the volume of the tetrahedron and the length of the height lowered from the vertex D.

9.13 Find out the orientation of the triple vectors:

1) $\vec{a}=\vec{i}-\vec{j}+2\vec{k}; \vec{b}=3\vec{i}+4\vec{j}+\vec{k}, \vec{c}=-2\vec{i}+3\vec{j}-\vec{k}$;

2) $\vec{a}=5\vec{i}+\vec{j}-2\vec{k}; \vec{b}=-3\vec{i}+2\vec{k}, \vec{c}=2\vec{i}+\vec{j}-\vec{k}$.

9.14 Find the height length of the parallelepiped built on the vectors $\vec{a}=\vec{i}-5\vec{j}+\vec{k}, \vec{b}=4\vec{i}+2\vec{k}, \vec{c}=\vec{i}-\vec{j}-\vec{k}$, if the basis is the parallelogram built on the vectors $\vec{a}$ and $\vec{b}$.

Homework

9.15 Find the vector $\vec{c}$, orthogonal to the vectors $\vec{a}=(2;-3;1)$ and $\vec{b}=(1;-2;3)$ and satisfying the condition $(\vec{c},\vec{i}+2\vec{j}-7\vec{k})=10$.

9.16 Calculate the area of the parallelogram constructed on vectors $\vec{a}=(0;-1;1)$ and $\vec{b}=(1;1;1)$.

9.17 Calculate the sine of the angle formed by the vectors $\vec{a}=(2;-2;1)$ and $\vec{b}=(2;3;6)$.

9.18 Set whether vectors are complanar $\vec{a},\vec{b},\vec{c}$, if $\vec{a}=(2;3;-1)$, $\vec{b}=(1;-1;3), \vec{c}=(1;9;-11)$.

9.19 Do the dots $A(5;5;4), B(3;8;4), C(3;5;10), D(5;8;2)$ belong to the same plane?

9.20 Find out whether the right or left triple form the next three vectors $\vec{a}=(3;4;0)$, $\vec{b}=(0;-4;1)$, $\vec{c}(0;2;5)$.

9.21 Calculate the volume of the tetrahedron *ABCD* and the length of the height omitted from the point *D* to the base of *ABC*, if the coordinates of its vertices are known $A(0,0,1), B(-3,2,3), C(2,-1,3), D(1,3,8)$.

9.22 Three forces are given $\vec{F}_1=(2;-1;3), \vec{F}_2=(3;2;-1), \vec{F}_3=(-4;1;-3)$, attached to the point $C=(-1;4;-2)$. Determine the magnitude and guide cosines of the angular momentum of equal force relative to the point $A=(2;3;-1)$.

Answers: 9.1 1) 12**; 2)** 24**; 3)** 48. **9.2 1)** (5;1,7)**; 2)** (10; 2; 14)**; 3)** (20; 4; 28). **9.3** $\vec{c}=4\vec{i}-5\vec{j}+2\vec{k}$.

9.4 $\sqrt{\frac{23}{185}}$. **9.5** 2; 5. **9.6** The area of the parallelogram, the sides of which are the diagonals of a given parallelogram, is equal to twice the area of this parallelogram.

9.7 **1)** $-4\vec{i}-2\vec{j}+2\vec{k}$; **2)** $2\vec{j}+2\vec{k}$. **9.9** 15; $\cos\alpha=\frac{2}{3}; \cos\beta=-\frac{2}{15}; \cos\gamma=-\frac{11}{15}$. **9.10** **a)** ea; **b)** **нfт**. **9.12** $\frac{154}{3};11$. **9.13** **1)** right three; **2)** left three. **9.14** $\frac{16}{3\sqrt{14}}$. **9.15** $\vec{c}=(7,5,1)$. **9.16** $\sqrt{6}$. **9.17** $\sin\varphi=\frac{5\sqrt{17}}{21}$. **9.18** Coplanar. **9.19** No, they don't. **9.20** Left.

9.21 $\frac{29}{6}$, $\frac{29}{\sqrt{137}}$. **9.22** $\sqrt{66}; \cos\alpha = -\frac{1}{\sqrt{66}}; \cos\beta = \frac{4}{\sqrt{66}}; \cos\gamma = \frac{7}{\sqrt{66}}$.

Lesson 10. A line on a plane

Classroom assignments

10.1 Write the equation of a straight line passing through the point $A(-1;\ 2)$ perpendicular to the vector $\overrightarrow{M_1M_2}$, where $M_1(2;-7)$, $M_2(3;\ 2)$.

10.2 Write canonical and parametric equations of a line passing through the point $A(3;-2)$ in parallel: 1) vector $\vec{S}(1;\ 5)$; 2) axes Oy.

10.3 Write the equation of a straight line passing through the point $A(-1;8)$ and forming an angle with the abscissa axis equals to $\frac{3\pi}{4}$.

10.4 Write the equation of a line passing through the points $M_1(2;1), M_2(4;5)$ and to find the point of its intersection with coordinate axes.

10.5 Find the equation of a line passing through the point $M_0(4;3)$ that is the base of a perpendicular lowered from the origin to this line.

10.6 At what value A is the straight line $Ax+4y-13=0$ forms an angle $\alpha = 45°$ with the *Ox* axis?

10.7 The vertices of the triangle are given $A(2;-3), B(4;5), C(-3;4)$. To find: 1) the equation of the side *AB*; 2) equation of the median drawn from vertex *C*; 3) equation of height drawn from the vertex *C*.

10.8 Write the equation of a straight line, parallel bisectrix of the second coordinate angle and cutting off a segment of 3 on the Oy axis.

10.9 Find the equation of a straight line passing through a point $A(2;-3)$: 1) parallel to the straight line $y=2x-9$; 2) perpendicular to the straight line $x+3y-2=0$.

10.10 What is the mutual arrangement of two straight lines, the angular coefficients of which are equal to – 2,5 and – 0,4?

10.11 Find the distance from a point $M(-1;2)$ to the next straight line:

1) $\begin{cases} x=-1+t, \\ y=2+3t, \end{cases}$ 2) $\begin{cases} x=5+2t, \\ y=-3-3t. \end{cases}$

10.12 Which lines of a given pair intersect, parallel, or coincide? If the lines intersect, find the coordinates of the point of their intersection:

1) $2x+y-1=0$ and $x-3y-2=0$; 2) $2x+6y=2$ and $x+3y-1=0$;

3) $-x-y=3$ and $3x+3y+1=0$; 4) $\frac{x-1}{2}=\frac{y+1}{-1}$ and $\frac{x-2}{-1}=\frac{y-2}{1}$.

10.13 Find the distance between the straight lines $12x-5y-26=0$ and $12x-5y+13=0$.

10.14 Find the point projection for the point $A(2;6)$ on the straight line $3x+4y-5=0$.

Homework

10.15 Find the equation of a line passing through the intersection point of a line $3x-2y-7=0$ and $x+3y-6=0$ and cutting off a segment of 3 on the abscissa axis.

10.16 Find the intersection point O of quadrilateral diagonals $ABCD$, if $A(-1;-3), B(3;5), C(5;2), D(3;-5)$.

10.17 The vertices of the triangle are given ABC: $A(1;2)$, $B(2;-2)$, $C(6;1)$. To find:

1) the equation of the side AB;

2) the equation of the height CH;

3) the equation of the median AM;

4) the equation of a straight line passing through the vertex C parallel to the side AB;

5) the distance from the point C to the straight line AB.

10.18 Find equations of perpendiculars to the line $3x+5y-15=0$ drawn through the intersection points of a given line with coordinate axes.

10.19 Find the equation of a straight line passing through the point $A(-2;3)$ and constituting with an angle axis Ox: a) 45º; b) 90º; c) 0º.

10.20 Find a point B that is symmetric to the point $A(8;12)$ relative to the straight line $x-2y+6=0$.

10.21 Find one of the angles between the straight lines:

1) $2x+3y-5=0$ and $x-3y-7=0$; 2) $\begin{cases} x=4 \\ y=t+7 \end{cases}$ and $\begin{cases} x=3t-1 \\ y=\sqrt{3}t+2 \end{cases}$.

Answers: **10.1** $x+9y-17=0$. **10.2** **1)** $\frac{x-3}{1}=\frac{y+2}{5}$, $\begin{cases}x=3+t\\ y=-2+5t\end{cases}$**;** **2)** $\frac{x-3}{0}=\frac{y+2}{1}$, $x=3$. **10.3** $x+y-7=0$. **10.4** $2x-y-3=0;\ (0;-3),\ (1{,}5;0)$. **10.5** $4x+3y-25=0$. **10.6** –4. **10.7** **1)** $4x-y-11=0$; **2)** $x+2y-5=0$; **3)** $x+4y-13=0$. **10.8** $y=-x+3$. **10.9** **1)** $2x-y-7=0$; **2)** $3x-y-9=0$. **10.10** Intersect. **10.11** **1)** 0; **2)** $\frac{8}{\sqrt{13}}$. **10.12 1)** $\left(\frac{5}{7};-\frac{3}{7}\right)$; **2)** match; **3)** parallel; **4)** $(9;-5)$. **10.13** 3. **10.14** $(-1,2)$. **10.15** $x=3$. **10.16** $O(3;1/3)$. **10.17** **1)** $\frac{x-1}{1}=\frac{y-2}{-4}$**;** **2)** $x-4y-2=0$**;** **3)** $5x+6y-17=0$**;** **4)** $4x+y-25=0$**;** **5)** $\frac{19}{\sqrt{17}}$. **10.18** $5x-3y-25=0,\ 5x-3y+9=0$. **10.19** **1)** $x-y+5=0$; **2)** $x+2=0$; **3)** $y-3=0$. **10.20** $B(12;\ 4)$. **10.21 1)** $\arccos\frac{7}{\sqrt{130}}$; **2)** $\frac{\pi}{3}=60^\circ$.

Lesson 11. The plane

Classroom assignments

11.1 Let us consider the next points $M_1(3;-1;2)$ and $M_2(4;-2;-1)$. Find the equation of the plane passing through the point M_1 perpendicular to the vector $\overrightarrow{M_1M_2}$.

11.2 Find the equation for a plane passing through three points: **1)** $M_1(3;-1;2)$, $M_2(4;-1;-1)$ and $M_3(2;0;2)$; **2)** $M_1(1;3;4)$, $M_2(3;0;2)$ and $M_3(2;5;7)$.

11.3 Specify the features in the location relative to the coordinate system *Oxyz* of the plane specified by the equation: **1)** $3y+2z-1=0$; **2)** $2x+y-5z=0$;

3) $2x-y-1=0$; **4)** $2x+y=0$; **5)** $x+z=0$; **6)** $3y-4z=0$; **7)** $2x+3=0$; **8)** $z+4=0$; **9)** $y=0$.

11.4 Find the lengths of the segments cut off on the coordinate axes by the plane $3x-2y+z-6=0$.

11.5 Find an equation for a plane passing through the point $M_0(1;-1;0)$ parallel to the next vectors: **1)** $\vec{a}=(0;2;3)$ and $\vec{b}=(-1;4;2)$; **2)** $\bar{s}_1=(2;-1;3)$ and $\bar{s}_2=(3;0;1)$.

11.6 Find an equation for a plane passing through the point $M_0(1;-3;-2)$ in parallel: **1)** plane $3x-2y+4z-3=0$; **2)** plane *Oyz*.

11.7 Find the equation of the plane passing through the point $M(1;\,0;\,-2)$ perpendicular to the planes $x-2y+z+5=0$ and $2x-y+3z-1=0$.

11.8 Find the angle between the planes: **1)** $x+4y-z+1=0$ and $x+y-z-3=0$; **2)** $x+2y-z+5=0$ and $2x-y+z-3=0$.

11.9 Given a pyramid with vertices $A(2;2;-3)$, $B(3;1;1)$, $C(-1;0;5)$, $D(4;-2;-3)$. Find the length of the height lowered from vertex *D* to the plane *ABC*.

11.10 Determine which of the following pairs of planes intersect, parallel or coincide:

1) $x-y+3z+1=0$ and $2x-y+5z-2=0$;

2) $3x+2y-z+2=0$ and $6x+4y-2z+1=0$;

3) $2x+6y+2z-4=0$ and $3x+9y+3z-6=0$.

11.11 Find the distance between the planes $2x-3y+6z-21=0$ and $4x-6y+12z+35=0$.

Homework

11.12 Find the equation of the plane passing through the point M perpendicular to the vector $\vec{n}$ for the next cases: **1)** $M(3;5;-1)$; $\vec{n}(13;2;1)$; **2)** $M(2;0;0)$; $\vec{n}(0;7;0)$; **3)** $M(0;3;-1)$; $\vec{n}=\overrightarrow{M_1M_2}$, where $M_1(1;-1;0)$, $M_2(3;0;2)$.

11.13 ~~Make~~ the equation of the plane passing through the point $M(-1;2;3)$ parallel to the plane passing through the points $M_1(1;0;-2)$, $M_2(3;4;5)$, $M_3(-1;2;0)$.

11.14 Determine at what value of the parameter α the plane $\alpha x+(2\alpha-1)y+z-5=0$:

1) parallel to the plane $2x+3y+z-4=0$;

2) parallel to the plane $y-z+7=0$;

3) perpendicular to the plane $3x+y-z=0$;

4) perpendicular to the plane *Oxz*.

11.15 ~~Make up~~ the equation of the plane passing through the points $M_1(1;2;3)$ and $M_2(2;1;1)$ perpendicular to the plane $3x+4y+z-6=0$.

11.16 Find the distance from the point $M(2;1;1)$ to the plane $x+y-z+1=0$.

11.17 Find the intersection point of the planes $x+y+z-6=0$, $2x-y+z-3=0$, $x+2y-z-2=0$.

Answers: **11.1** $x-y-3z+2=0$. **11.2** **1)** $3x+3y+z-8=0$**;** **2)** $5x+8y-7z-1=0$. **11.3** **1)** parallel to the axis *Ox***;** **2)** passes through the origin coordinates; **3)** parallel to the axis *Oz***;** **4)** passes through the axis *Oz***; 5)** passes through the axis *Oy***; 6)** passes through the axis *Ox***; 7)** parallel to the plane *Oyz***; 8)** parallel to the plane *Oxy***;** **9)** coincides with the plane *Oxz*. **11.4** 2; 3; 6. **11.5** **1)** $8x+3y-2z-5=0$; **2)** $x-7y-3z-8=0$. **11.6** **1)** $3x-2y+4z-1=0$; **2)** $x=1$. **11.7** $5x+y-3z-11=0$. **11.8** $\arccos\frac{\sqrt{6}}{3}$. **11.9** **1)** intersect; **2)** parallel; **3)** coincides. **11.10** $h=5$. **11.11** 5,5. **11.12** **1)** $13x+2y+z-48=0$; **2)** $y=0$; **3)** $2x+y+2z-1=0$. **11.13** $x+3y-2z+1=0$. **11.14 1)** $\alpha=2$; **2)** $\alpha=0$; **3)** $\alpha=0{,}4$**; 4)** $\alpha=0{,}5$. **11.15** $x-y+z-2=0$. **11.16** $\sqrt{3}$. **11.17** $(1;2;3)$

.

Lesson 12. Line in space. Lines and planes in space

Classroom assignments

12.1 ~~Make up t~~he canonical and parametric equations of a line passing through the point $M_0(2;0;-3)$ parallel to the vector $\vec{a}=(2;-3;5)$.

12.2 ~~Make up~~ the ~~canonical~~ and parametric equations of the line: **1)** $\begin{cases} x+y+2z-3=0, \\ x-y+z-1=0; \end{cases}$ **2)** $\begin{cases} 2x+y+z-1=0, \\ 3x+2y+z-2=0. \end{cases}$

12.3 Find the angle between the straight next lines: $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z+3}{-1}$ and $\dfrac{x+1}{-3}=\dfrac{y}{4}=\dfrac{z-10}{6}$.

12.4 At what values *a* are the lines $\dfrac{x-1}{a}=\dfrac{y-1}{1}=\dfrac{z-(a-2)^2}{a}$ and $\dfrac{x}{1}=\dfrac{y}{a}=\dfrac{z}{1}$:

1) intersect; 2) interbreed; 3) parallel; 4) coincide?

12.5 Make equations for the sides of a triangle with vertices in points $A(-3;2;1)$; $B(1;-1;0)$; $C(2;3;-5)$.

12.6 Find the equations of a line passing through the point $M(2;-5;4)$ parallel to the straight line $\dfrac{x+1}{2}=\dfrac{y-2}{-3}=\dfrac{z+5}{4}$.

12.7 Find out the relative location of the line and the plane: $\dfrac{x+1}{2}=\dfrac{y-3}{4}=\dfrac{z}{5}$ and $x-3y+2z-5=0$.

12.8 Find the canonical equation of a straight line passing through the point $M(2;-1;3)$ perpendicular to the plane $3x-y+2z-4=0$.

12.9 Find the angle between the straight line and the plane:

1) $\frac{x-1}{3}=\frac{y+2}{2}=\frac{z}{-6}$ and $4x+4y-7z+1=0$; **2)** $\begin{cases} x+4y-2z+7=0, \\ 3x+7y-2z=0 \end{cases}$ and $3x+y-z+1=0$.

12.10 Find the equation of a plane passing through a point $M(2;0;–3)$ parallel to the next straight lines: $\frac{x-2}{3}=\frac{y+1}{1}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$.

12.11 Find the coordinates of the point of intersection of the line $\frac{x-1}{2}=\frac{y+2}{1}=\frac{z-2}{1}$ with plane $3x-y+2z+5=0$.

12.12 Find the point projection $A(3;-1;4)$ on the plane $2x+y-z+5=0$.

12.13 Find the point A, symmetric to the point $P(6;–5;5)$ relative to the plane $2x-y+z-4=0$.

12.14 Find the point projection $A(2;3;1)$ to the line $\frac{x+7}{1}=\frac{y+2}{2}=\frac{z+2}{3}$ and the distance from that point to that line.

Homework

12.15 Find the equation of a line passing through the point $M(4;–3;2)$: **1)** parallel to the axis *Ox*; **2)** parallel to the axis *Oz*; **3)** perpendicular to the plane $x-3y+2z-5=0$; **4)** perpendicular to the plane *Oxz*.

12.16 Calculate the angle between the line $\begin{cases} x-2y+3=0 \\ 3y+z-1=0 \end{cases}$ and plane $2x+3y-z+1=0$.

12.17 Find the equations of the perpendicular drawn from the point $A(3;-5;1)$ to the next plane: **1)** $2x-y+5z+3=0$; **2)** $3x-2z+4=0$; **3)** $y-1=0$.

12.18 Do the next straight lines intersect?

1) $\frac{x+2}{-1}=\frac{y-3}{2}=\frac{z-4}{3}$ and $\frac{x}{3}=\frac{y+4}{2}=\frac{z-3}{5}$; **2)** $\begin{cases} x+3y-4z+7=0, \\ 3x+y+2z-5=0, \end{cases}$ and $\begin{cases} x-y+3z-6=0, \\ 2x+y-z+3=0 \end{cases}$?

12.19 Find the parametric equations of the median of a triangle with vertices $A(3;6;-7), B(-5;1;-4), C(0;2;3)$, drawn from the vertex C.

12.20 Find the coordinates of the point Q, symmetric to the point $P(-3;1;-9)$ relative to the plane $4x-3y-z-7=0$.

12.21 Find the coordinates of a point Q, symmetric to the point $P(2;-5;7)$ relative to a line passing through the points $M_1(5;4;6)$ and $M_2(-2;-17;-8)$.

12.22 Find the angle between the straight lines:

1) $\frac{x+2}{3}=\frac{y}{4}=\frac{z+1}{0}$ and axis Ox; **2)** $\begin{cases} x+y=0, \\ x-y=0, \end{cases}$ and $\begin{cases} y+z=0, \\ y-z+2=0. \end{cases}$

Answers: **12.1** $\frac{x-2}{2}=\frac{y}{-3}=\frac{z+3}{5}; \begin{cases} x=2+2t, \\ y=-3t, \\ z=-3+5t. \end{cases}$ **12.2** **1)** $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z}{-2}; \begin{cases} x=2+3t, \\ y=1+t, \\ z=-2t. \end{cases}$ **2)** $\frac{x+1}{-1}=\frac{y-2}{1}=\frac{z-1}{1}; \begin{cases} x=-1-t, \\ y=2+t, \\ z=1+t. \end{cases}$. **12.3** $\frac{\pi}{2}$. **12.4** **1)** $a=3$; **2)** $a\neq\pm1; a\neq3$; **3)** $a=-1$; **4)** $a=1$. **12.5** $\frac{x+3}{4}=\frac{y-2}{-3}=\frac{z-1}{-1}$; $\frac{x+3}{5}=\frac{y-2}{1}=\frac{z-1}{-6}$; $\frac{x-1}{1}=\frac{y+1}{4}=\frac{z}{-5}$. **12.6** $\frac{x-2}{2}=\frac{y+5}{-3}=\frac{z-4}{4}$. **12.7** A straight line parallel to a plane. **12.8** $\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-3}{2}$. **12.9** **1)** $\arcsin\left(\frac{62}{63}\right)$; **2)** $\arcsin\left(\frac{19}{11\sqrt{7}}\right)$. **12.10** $x+2y-5z-17=0$. **12.11** $(-3;-4;0)$. **12.12.** $(1;-2;5)$.. **12.13** $A(-2;7;1)$.

12.14 $(-5;2;4)$; $\sqrt{59}$. **12.15** **1)** $\frac{x-4}{1}=\frac{y+3}{0}=\frac{z-2}{0}$; **2)** $\frac{x-4}{0}=\frac{y+3}{0}=\frac{z-2}{1}$; **3)** $\frac{x-4}{1}=\frac{y+3}{-3}=\frac{z-2}{2}$;

4) $\frac{x-4}{0}=\frac{y+3}{1}=\frac{z-2}{0}$. **12.16** $\sin\varphi=\frac{5}{7}$; $\varphi\approx 45^\circ 36'$. **12.17** **1)** $\frac{x-3}{2}=\frac{y+5}{-1}=\frac{z-1}{5}$**;**

2) $\frac{x-3}{3}=\frac{y+5}{0}=\frac{z-1}{-2}$ **;** **3)** $\frac{x-3}{0}=\frac{y+5}{1}=\frac{z-1}{0}$. **12.18** **1)** no; **2)** yes. **12.19** $\begin{cases} x=2t, \\ y=-3t+2, \\ z=17t+3. \end{cases}$

12.20 $Q(1;-2;-10)$. **12.21** $Q(4;1;-3)$. **12.22. 1)** $\cos\varphi=\frac{3}{5}$; **2)** $\cos\varphi=\frac{6}{\sqrt{61}}$.

Lesson 13. Second-order curves on a plane

Classroom assignments

13.1 For the following ellipses and hyperboles, find: a) semi-axes; b) the distance between the focuses; c) eccentricity ε; d) focal points; e) vertex coordinates; (e) for hyperbolas, make up the asymptote equations.

1) $\frac{x^2}{16}+\frac{y^2}{25}=1$; **2)** $\frac{x^2}{25}+\frac{y^2}{16}=1$; **3)** $\frac{x^2}{144}-\frac{y^2}{25}=-1$; **4)** $\frac{x^2}{144}-\frac{y^2}{25}=1$.

13.2 Find the equation of an ellipse whose foci lie on the abscissa axis and are symmetric with respect to the origin, if:

1) its semi-axes are 1 and 7;

2) the distance between the focuses is 8 and the semi-minor axis is 3;

3) semi-major axis is 5 and the point $M_0(3;-2;4)$ belongs to the ellipse.

13.3 Find the equation of an ellipse whose foci lie on the ordinate axis and are symmetric with respect to the origin, if:

1) its semi-axes are 2 and 5;

2) the focal length is 12 and the semi-major axis is 13;

3) the minor axis is 10 and the eccentricity $\varepsilon=\frac{12}{13}$.

13.4 Find the hyperbole equation, if:

1) her focuses are at points $F_1(7;0)$, $F_2(-7;0)$, and the real semi-axis is 5;

2) hyperbole passes through a point $M_0\left(6;-2{,}5\sqrt{3}\right)$, and its vertices are at points $A_1(-4;0)$, $A_2(4;0)$.

13.5 Find the equation of a hyperbola whose foci lie on the ordinate axis and are symmetric with respect to the origin, if:

1) its real and imaginary semi-axes are 11 and 4 respectively.;

2) the distance between the focuses is 10 and the eccentricity $\varepsilon = \frac{5}{3}$;

3) equation of one of the asymptotes $y = \frac{3}{4}x$, and the actual semi-axis is 6.

13.6 Find the canonical parabola equation, if:

1) its vertex coincides with the origin, and the focus is at a point $F(2;0)$;

2) the branches are directed upwards, and the parameter is 4;

3) headmistress equation $y = 3$, and the focus is at the point $F(0;-3)$;

4) its apex coincides with the origin, the parabola passes through the point $M_0(9;-6)$ and the abscissa axis is the axis of the parabola.

13.7 Determine the type and location of the line of the second order, plot it:

1) $9x^2 + 4y^2 - 18x + 16y - 11 = 0$;

2) $9x^2 - 16y^2 + 54x + 64y - 127 = 0$;

3) $x^2 - 10x - 8y + 49 = 0$;

4) $y^2 - 2x - 4y - 2 = 0$;

5) $4x^2 - 9y^2 - 16x - 18y + 7 = 0$;

6) $x^2 + 4y^2 + 4x - 32y + 68 = 0$.

Homework

13.8 Write the canonical ellipse equation, if it is known that:

1) the focal length is 8, the semi-minor axis is equal to 3;

2) the semi-minor axis is 6, the eccentricity is equal to 4/5.

13.9 find focal coordinates and ellipse eccentricity $4x^2+y^2=4$.

13.10 Find the canonical equation of an ellipse passing through points $M_1\left(\frac{3\sqrt{3}}{2};-1\right)$ and $M_2\left(-1;\frac{4\sqrt{2}}{3}\right)$, and to find its eccentricity.

13.11 Find the hyperbola equation, if its asymptotes are given by the equations $x\pm 2y=0$, and the distance between the vertices lying on the *Ox* axis is 4.

13.12 Compose the canonical hyperbola equation, if it is known that:

1) the distance between the focuses is 30 and the distance between the vertices is 24;

2) the real semi-axis is 4 and the hyperbola passes through the point $M(2;4\sqrt{2})$.

13.13 Find the equation of a hyperbola whose vertices are in the foci and the foci are in the vertices of the ellipse $6x^2+5y^2=30$.

13.14 Find the canonical parabola equation, if it is known that:

1) the parabola has a focus $F(0;2)$ and a vertex at the point $O(0;0)$;

2) the parabola is symmetrical with respect to the *Ox* axis and passes through the point $M(4;-2)$.

13.15 Find the length of the common chord of the parabola $y=2x^2$ and circles $x^2+y^2=5$.

13.16 Write the equation of a parabola passing through the points $(0;\,0)$ and $(-2;\,4)$, if parabola is symmetrical relative to: **1)** axis Ox; **2)** axis Oy .

13.17 Write the canonical parabola equation whose foci coincide with the foci of the hyperbola $x^2-y^2=8$.

13.18 Which figure corresponds to each of these equations? In the case of a non-empty set depict it in the coordinate system *Oxy*:

1) $x^2+y^2-4x+6y+4=0$;

2) $3x^2-4y^2-12x-8y+20=0$;

3) $y^2-3x-4y+10=0$;

4) $2x^2+3y^2+6x+6y+25=0$.

5) $4x^2+25y^2+4x-10y-8=0$;

6) $x^2-y^2+2x-2y=0$;

7) $x^2-6x+2y+11=0$.

Answers: **13.1** **1)** **a)** $a=4; b=5$; **b)** $2c=6$; **c)** $\varepsilon=\frac{3}{5}$; **d)** $F_1(0;-3), F_2(0;3)$; **e)** $(4;0), (-4;0), (0;5), (0;-5)$;

2) a) $a=5; b=4$; **b)** $2c=6$; **c)** $\varepsilon=\frac{3}{5}$; **d)** $F_1(3;0), F_2(-3;0)$; **e)** $(5;0), (-5;0), (0;4), (0;-4)$;

3) **a)** $a=12; b=5$; **b)** $2c=26$; **c)** $\varepsilon=\frac{13}{5}$; **d)** $F_1(0;-13), F_2(0;13)$; **e)** $A_1(0;-5), A_2(0;5)$; **f)** $y=\pm\frac{5}{12}x$; **4)** **a)** $a=12; b=5$; **b)** $2c=26$; **c)** $\varepsilon=\frac{13}{12}$; **d)** $F_1(13;0), F_2(-13;0)$; **e)** $A_1(-12;0), A_2(12;0)$; **f)** $y=\pm\frac{5}{12}x$.

13.2 1) $\frac{x^2}{49}+\frac{y^2}{1}=1$; **2)** $\frac{x^2}{25}+\frac{y^2}{9}=1$; **3)** $\frac{x^2}{25}+\frac{y^2}{9}=1$.

13.3 1) $\frac{x^2}{4}+\frac{y^2}{25}=1$; **2)** $\frac{x^2}{133}+\frac{y^2}{169}=1$; **3)** $\frac{x^2}{25}+\frac{y^2}{169}=1$.

13.4 1) $\frac{x^2}{25}-\frac{y^2}{24}=1$; **2)** $\frac{x^2}{16}-\frac{y^2}{15}=1$.

13.5 1) $\frac{y^2}{121}-\frac{x^2}{16}=1$; **2)** $\frac{y^2}{9}-\frac{x^2}{16}=1$; **3)** $\frac{y^2}{36}-\frac{x^2}{64}=1$.

13.6 1) $y^2=8x$; **2)** $x^2=8y$; **3)** $x^2=-12y$; **4)** $y^2=4x$.

13.7 Find out which figure corresponds to each of these equations, and (in the case of a non-empty set) depict it in a coordinate system is at the point O'.

1) ellipse $\frac{X^2}{4}+\frac{Y^2}{9}=1, O'(1;-2)$; **2)** hyperbola $\frac{X^2}{16}-\frac{Y^2}{9}=1, O'(-3;2)$;

3) parabola $X^2=8Y, O'(5;3)$; **4)** parabola $Y^2=2X, O'(-3;2)$;

5) the pair of intersecting lines $2x-3y-7=0$ and $2x+3y-1=0$; **6)** the point $O'(-2;4)$.

13.8 1) $\frac{x^2}{25}+\frac{y^2}{9}=1$; $\frac{x^2}{9}+\frac{y^2}{25}=1$; **2)** $\frac{x^2}{36}+\frac{y^2}{100}=1$; $\frac{x^2}{100}+\frac{y^2}{36}=1$.

13.9 $F_1(0,-\sqrt{3})$, $F_2(0,\sqrt{3})$, $\varepsilon=\frac{\sqrt{3}}{2}$. **13.10** $\frac{x^2}{9}+\frac{y^2}{4}=1$; $\varepsilon=\frac{\sqrt{5}}{3}$. **13.11** $\frac{x^2}{4}-\frac{y^2}{1}=1$.

13.12. **1)** $\frac{x^2}{144}-\frac{y^2}{81}=1$; $\frac{y^2}{144}-\frac{x^2}{81}=1$; **2)** $\frac{y^2}{16}-\frac{x^2}{4}=1$. **13.13** $\frac{y^2}{1}-\frac{x^2}{5}=1$. **13.14** **1)** $x^2=8y$;

2) $y^2=x$. **13.15** 2. **13.16 1)** $y^2=-8x$; **2)** $y=x^2$. **13.17** $y^2=\pm 16x$.

13.18 1) circles $(x-2)^2+(y+3)^2=12$**; 2)** hyperbola $\frac{(y+1)^2}{3}-\frac{(x-2)^2}{4}=1$;

3) parabola $(y-2)^2=3(x-2)$; **4)** empty set; **5)** ellipse $\frac{(x+0,5)^2}{2,5}+\frac{(y-0,2)^2}{0,4}=1$; **6)** a pair of intersecting lines $x+y+2=0; x-y=0$; **7)** parabola $(x-3)^2=-2(y+1)$.

Lesson 14. Second-order surfaces

Classroom assignments

14.1 Determine the appearance of surfaces and their location relative to the coordinate axes:

1) $\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1$; **2)** $\frac{x^2}{16}+\frac{y^2}{25}-\frac{z^2}{100}=-1$; **3)** $\frac{x^2}{16}-\frac{y^2}{25}-\frac{z^2}{100}=1$;

4) $\frac{x^2}{25}-\frac{y^2}{64}+\frac{z^2}{49}=-1$; **5)** $\frac{x^2}{16}+\frac{y^2}{4}=2z$; **6)** $\frac{x^2}{9}-\frac{y^2}{16}-\frac{z^2}{25}=0$;

7) $\frac{x^2}{16}-\frac{y^2}{25}-\frac{z^2}{100}=-1$; **8)** $\frac{x^2}{16}+\frac{z^2}{25}=-2x$; **9)** $\frac{x^2}{16}+\frac{z^2}{100}=1$;

10) $\frac{x^2}{16}-\frac{z^2}{25}=1$; **11)** $z^2=18x$.

14.2 Find the canonical form for the equation of the second order, using the parallel transport transformation, to determine the appearance of the surface and its location relative to the new coordinate system:

1) $9x^2+4y^2+4z^2-18x+16z-11=0$; **2)** $9x^2+4y^2-4z^2-18x-16z-43=0$;

3) $9x^2-4y^2+4z^2+18x+16z+25=0$; **4)** $9y^2+4z^2=36x+72$;

5) $x^2+y^2+6x-4y+12=0$; **6)** $y^2=4x+16$; **7)** $x^2+y^2+z^2+6x-4y+2z-10=0$.

14.3 Plot a body bounded by surfaces:

1) $x^2+y^2=4;\ z=0;\ z=1;\ y=x;\ y=x\sqrt{3}$, located in the first octant;

2) $x^2+y^2=2x;\ z=0;\ z=x^2+y^2$; **3)** $z=x^2+y^2+1;\ x=0;\ y=0;\ z=0;\ x=4;\ y=4$;

4) $z=x^2-y^2;\ z=0;\ x=3$; **5)** $x^2+y^2+z^2=9;\ x^2+y^2=3a$.

Homework

14.4 Determine the type of surface and build it:

1) $x^2+y^2+z^2-3x+5y-4z=0$; **2)** $x=y^2+2z^2$; **3)** $2x^2-y^2+z^2=4$;

4) $2x^2-y^2+3z^2=0$; **5)** $z^2=4x$; **6)** $x^2+z^2=5$;

7) $x^2+y^2+z^2=2z$; **8)** $x^2+3z^2-8x+18z+34=0$; **9)** $5x^2+y^2+10x-6y-10z+14=0$;

Answers: 14.1 1) ellipsoid; 2) a bicavitary hyperboloid elongated along the *Oz* axis; 3) a bicavitary hyperboloid elongated along the *Ox* axis; 4) a bicavitary hyperboloid elongated along the *Oy* axis; 5) an elliptical paraboloid elongated in the positive direction of the *Oz* axis; 6) a cone elongated along the *Ox* axis; 7) a single-cavity hyperboloid elongated along the *Ox* axis; 8) an elliptical paraboloid elongated in the negative direction of the *Ox* axis; 9) an elliptical cylinder forming parallel to the *Oy* axis; 10) hyperbolic cylinder forming parallel to the *Oy* axis; 11) a parabolic cylinder forming parallel to the *Oy* axis.

14.2 In all tasks, the new coordinate axes *OX*, *OY*, *OZ* are aligned with the old one, the origin of the coordinates of the new coordinate system is at the point *O′*.

1) ellipsoid $\frac{X^2}{4}+\frac{Y^2}{9}+\frac{Z^2}{9}=1;\ O'(1;0;-2)$;

2) unicavitary hyperboloid $\frac{X^2}{4}+\frac{Y^2}{9}-\frac{Z^2}{9}=1$, elongated along the axis *OZ*, $O'(1;0;-2)$;

3) second-order cone $9X^2-4Y^2+4Z^2=0$, elongated along the axis *OY*, $O'(-1;0;-2)$;

4) elliptical paraboloid $\frac{Y^2}{4}+\frac{Z^2}{9}=X$, elongated in the positive direction of the axis *OX*, $O'(-2;0;0)$;

5) elliptical cylinder (circular) $X^2+Y^2=1$, forming parallel to the axe *OZ*, $O'(-3;2;0)$;

6) parabolic cylinder $Y^2=4X$, forming parallel to the axe *OZ*, $O'(-4;0;0)$;

7) sphere $X^2+Y^2+Z^2=4$, $O'(-3;2;-1)$.

14.4 1) sphere $\left(x-\frac{3}{2}\right)^2+\left(y+\frac{5}{2}\right)^2+(z-2)^2=\frac{25}{2}$; **2)** elliptical paraboloid;

3) single-cavity hyperboloid; 4) conical surface; 5) parabolic cylinder; 6) circular cylinder;

7) sphere $x^2+y^2+(z-1)^2=1$; **8)** elliptic cylinder $\frac{(x-4)^2}{9}+\frac{(z+3)^2}{3}=1$; **9)** elliptical paraboloid $z=\frac{(x+1)^2}{2}+\frac{(y-3)^2}{10}$.

Lesson 15. Function. Sequence limit and function limit

Classroom assignments

15.1 Construct the feature definition areas:

1) $y=\sqrt{x^2-6x+5}$; **2)** $y=\arccos\frac{2x}{1+x}$; **3)** $y=\sqrt{25-x^2}+\lg\sin x$; **4)** $y=2^{x^2-2}$.

15.2 Check the next functions for parity or oddness:

1) $f(x)=x^4+5x^2$; **2)** $f(x)=x^2+x$; **3)** $f(x)=\dfrac{x}{2^x-1}$; **4)** $f(x)=\dfrac{e^x+1}{e^x-1}$.

15.3 Plot the graphs for the next functions:

1) $y=\dfrac{2x+3}{x-1}$; **2)** $y=|3x+4-x^2|$; **3)** $y=-2\sin(2x+2)$; **4)** $y=x\sin x$.

15.4 Calculate limits:

1) $\lim\limits_{x\to 1}\left(2x^2+2x-3\right)$; **2)** $\lim\limits_{x\to -3}(x+3)^2(x-1)$; **3)** $\lim\limits_{x\to 0}\dfrac{1-3x-x^2}{2x^2+x-3}$; **4)** $\lim\limits_{x\to 2}\dfrac{3x+1}{2-x}$;

5) $\lim\limits_{x\to\infty}\dfrac{4}{x+1}$; **6)** $\lim\limits_{x\to\infty}\dfrac{5x^2-3x+1}{3x^2+x-5}$; **7)** $\lim\limits_{x\to\infty}\dfrac{x^3+2x}{5-3x^2+4x^4}$; **8)** $\lim\limits_{x\to\infty}\dfrac{1-3x^2+x^3}{2x+1}$;

9) $\lim\limits_{n\to\infty}\left(\dfrac{2n-1}{5n+7}-\dfrac{1+2n^3}{2+5n^3}\right)$; **10)** $\lim\limits_{n\to\infty}\dfrac{(n+1)^3-(n-1)^3}{(n+1)^2+(n-1)^2}$; **11)** $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2+n+2}}{n+1}$;

12) $\lim\limits_{n\to\infty}\dfrac{n!}{(n+1)!-n!}$; **13)** $\lim\limits_{x\to 2}\dfrac{x^3-8}{4-x^2}$; **14)** $\lim\limits_{x\to 3}\dfrac{3x^2-27}{81-x^4}$; **15)** $\lim\limits_{x\to 4}\dfrac{2x^2-9x+4}{x^2+x-20}$;

16) $\lim\limits_{x\to 5}\dfrac{x^2-25}{x^2-6x+5}$; **17)** $\lim\limits_{x\to 1}\dfrac{4x^2+5x-9}{2x^2+3x-5}$; **18)** $\lim\limits_{x\to 1}\dfrac{\sqrt{x+8}-3}{x-1}$; **19)** $\lim\limits_{x\to 4}\dfrac{\sqrt{3x+4}-4}{\sqrt{x}-2}$;

20) $\lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3}{1-\sqrt{3-x}}$; **21)** $\lim\limits_{n\to\infty}\left(\dfrac{1+2+3+\ldots+n}{n+2}-\dfrac{n}{2}\right)$; **22)** $\lim\limits_{x\to\infty}\left(\sqrt{x+2}-\sqrt{x}\right)$;

23) $\lim\limits_{n\to\infty}\left(\sqrt{n^2-2n-1}-\sqrt{n^2-7n+3}\right)$; **24)** $\lim\limits_{x\to\infty}x^{3/2}\left(\sqrt{x^3+1}-\sqrt{x^3-1}\right)$;

25) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{16+x^2}-4}$; **26)** $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$; **27)** $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\sin x-\cos x}{\cos 2x}$; **28)** $\lim\limits_{n\to \infty}\dfrac{\sqrt{n}\sin n}{n+1}$;

29) $\lim\limits_{x\to 1}\dfrac{x^3-3x^2+2}{x^2-7x+6}$; **30)** $\lim\limits_{x\to 1}\dfrac{\sqrt{x}+\sqrt{x-1}-1}{\sqrt{x^2-1}}$; **31)** $\lim\limits_{n\to \infty}\dfrac{3^n+5^n}{3^n-5^n}$; **32)** $\lim\limits_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)$.

Homework

15.5 Find the limits for the specified functions:

1) $\lim\limits_{x\to \infty}\dfrac{2+4x^2+3x^3}{x^3-7x-10}$; **2)** $\lim\limits_{x\to \infty}\dfrac{7x^2+10x+20}{x^3-10x^2-1}$; **3)** $\lim\limits_{n\to \infty}\dfrac{(n+2)!+(n+1)!}{(n+3)!}$;

4) $\lim\limits_{n\to \infty}\dfrac{5^n-3}{5^{n+1}+2}$; **5)** $\lim\limits_{n\to \infty}\dfrac{1}{n^2}(1+2+3+...+n)$; **6)** $\lim\limits_{x\to 4}\dfrac{3x^2-10x-8}{16-x^2}$; **7)** $\lim\limits_{x\to 1}\dfrac{x^3-x^2+x-1}{x^2-4x+3}$;

8) $\lim\limits_{x\to 5}\dfrac{x^2-25}{\sqrt{x-1}-2}$; **9)** $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{\sqrt{x^2+9}-3}$; **10)** $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1-x}-1}{x}$;

11) $\lim\limits_{x\to \infty}\left(x\left(\sqrt{x^2+5}-\sqrt{x^2+1}\right)\right)$; **12)** $\lim\limits_{x\to 2}\left(\dfrac{1}{2-x}-\dfrac{3}{8-x^2}\right)$; **13)** $\lim\limits_{x\to -3}\dfrac{x^2+x-6}{\sqrt{7-3x}-4}$;

14) $\lim\limits_{x\to \infty}\left(x-\dfrac{x^3}{x^2+3x-2}\right)$; **15)** $\lim\limits_{x\to -5}\dfrac{\sqrt{3x+17}-\sqrt{2x+12}}{x^2+8x+15}$.

Answers: **15.1** **1)** $(-\infty;1]\cup[5;+\infty)$; **2)** $\left[-\dfrac{1}{3};1\right]$; **3)** $[-5;-\pi)\cup(0;\pi)$; **4)** $(-\infty;+\infty)$.

15.2 **1)** even; (2) neither even nor odd; 3) neither even nor odd; 4) odd.

15.4 **1)** 1; **2)** 0; **3)** $-\frac{1}{3}$; **4)** ∞; **5)** 0; **6)** $\frac{5}{3}$; **7)** 0; **8)** ∞; **9)** 0; **10)** 3; **11)** 0; **12)** 0; **13)** – 3; **14)** $-\frac{1}{6}$; **15)** $-\frac{2}{5}$; **16)** $\frac{5}{2}$; **17)** $\frac{13}{7}$; **18)** $\frac{1}{6}$; **19)** $\frac{3}{2}$; **20)** $\frac{1}{3}$; **21)** $-\frac{1}{6}$; **22)** 0; **23)** $\frac{5}{2}$; **24)** 1; **25)** 3; **26)** $\frac{2}{3}$; **27)** $-\frac{1}{\sqrt{2}}$; **28)** 0; **29)** $\frac{3}{5}$; **30)** $\frac{\sqrt{2}}{2}$; **31)** – 1; **32)** – 1.

15.5 **1)** 3; **2)** 0; **3)** 0; **4)** $\frac{1}{5}$; **5)** $\frac{1}{2}$; **6)** $-\frac{7}{4}$; **7)** – 1; **8)** 40; **9)** $\frac{3}{2}$; **10)** $-\frac{1}{3}$; **11)** 2; **12)** ∞; **13)** $\frac{40}{3}$; **14)** 3; **15)** $-\frac{\sqrt{2}}{8}$.

Lesson 16. First and Second Wonderful Limits

Classroom assignments

16.1 Calculate using the first wonderful limit:

1) $\lim\limits_{x\to 0}\frac{\sin 5x}{x}$; **2)** $\lim\limits_{x\to 0}\frac{x}{\sin 3x}$; **3)** $\lim\limits_{x\to 0}\frac{\operatorname{tg} 7x}{\sin 2x}$; **4)** $\lim\limits_{x\to 0}\frac{4x}{3\arcsin 2x}$; **5)** $\lim\limits_{x\to 0}\frac{\operatorname{arctg} 5x}{2x}$;

6) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$; **7)** $\lim\limits_{x\to 0}\frac{1-\cos 6x}{x\sin 3x}$; **8)** $\lim\limits_{x\to 0}\frac{\sin^2 3x}{\operatorname{tg}^2 5x\cdot\cos 2x}$; **9)** $\lim\limits_{x\to 0}\frac{\cos x-\cos 3x}{x^2}$;

10) $\lim\limits_{x\to 0}\frac{\operatorname{tg} x-\sin x}{x^3}$; **11)** $\lim\limits_{x\to 0}\frac{\sin 3x}{\sqrt{x+4}-2}$; **12)** $\lim\limits_{x\to 1}\frac{x^2-7x+6}{\sin(2(x-1))}$; **13)** $\lim\limits_{x\to -1}\frac{\operatorname{tg}(x+1)}{x^2-4x-5}$;

14) $\lim\limits_{x\to 0}\dfrac{\sqrt{2+\sin 3x}-\sqrt{2-\sin 3x}}{5\,\mathrm{tg}\,2x}$; **15)** $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\cos x}{\cos 2x}$; **16)** $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}-2\cos x}{\pi-4x}$;

17) $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos^3 2x}{4x\,\mathrm{tg}\,3x}$; **18)** $\lim\limits_{x\to\infty}\sqrt{x}\cdot\sin\dfrac{1}{x}$; **19)** $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\mathrm{ctg}\,x\right)$;

20) $\lim\limits_{x\to 0}\dfrac{3x\,\mathrm{arctg}5x}{\cos x-\cos 4x}$; **21)** $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\pi}{2}-x\right)\mathrm{tg}\,x$.

16.2 Find the limits given below using the second wonderful limit:

1) $\lim\limits_{x\to 0}(1+2x)^{1/x}$; **2)** $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{2x-1}\right)^{2x-3}$; **3)** $\lim\limits_{x\to\infty}\left(\dfrac{x-6}{x+5}\right)^{x+2}$; **4)** $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+3}\right)^{x}$;

5) $\lim\limits_{x\to 0}(1-7x)^{3/x}$; **6)** $\lim\limits_{x\to 0}\left(\dfrac{7x+3}{9x+3}\right)^{1/x}$; **7)** $\lim\limits_{x\to\infty}\left(\dfrac{2x+1}{6x-3}\right)^{x}$; **8)** $\lim\limits_{x\to\infty}((2x+1)(\ln(3x+1)-\ln(3x-2)))$;

9) $\lim\limits_{x\to\infty}\left(\dfrac{6-x}{7-x}\right)^{\frac{1-x^3}{x^2}}$; **10)** $\lim\limits_{x\to+\infty}\left(\dfrac{x}{1+x}\right)^{x}$; **11)** $\lim\limits_{x\to\infty}((x-4)(\ln(2-3x)-\ln(5-3x)))$;

12) $\lim\limits_{x\to 0}(1+\sin x)^{\mathrm{cosec}\,x}$; **13)** $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^{x}$; **14)** $\lim\limits_{x\to 0}\left(1+3\,\mathrm{tg}^2\,x\right)^{\mathrm{ctg}^2\,x}$;

15) $\lim\limits_{x\to 0}(1-4x)^{(1-x)/x}$; **16)** $\lim\limits_{x\to 0}\dfrac{\ln(1+7x)}{x}$; **17)** $\lim\limits_{x\to 1}(2x-1)^{2x/(x-1)}$.

16.3 Calculate the next limits:

1) $\lim\limits_{x\to 0}(\cos x)^{1/x^2}$; **2)** $\lim\limits_{x\to 0}\left(1+\operatorname{tg}^2\sqrt{x}\right)^{1/2x}$; **3)** $\lim\limits_{x\to 0}(1+x+x^2)^{1/\sin x}$; **4)** $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{3x}$;

5) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x}$; **6)** $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{3^x-1}$; **7)** $\lim\limits_{x\to 0}\dfrac{a^{2x}-1}{x}$.

Homework

16.4 Find the limits of the following functions:

1) $\lim\limits_{x\to 0}\dfrac{\sin 8x}{2\operatorname{arctg}3x}$; **2)** $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{3\sin x\operatorname{tg}3x}$; **3)** $\lim\limits_{x\to 0}\dfrac{\cos 3x-\cos x}{1-\sqrt{1-x^2}}$; **4)** $\lim\limits_{x\to 0}\dfrac{x\sin 4x}{\sqrt{1+\operatorname{tg}^2 2x}-\sqrt{1-\operatorname{tg}^2 2x}}$;

5) $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x^2\sin x}$; **6)** $\lim\limits_{x\to \frac{1}{2}}\dfrac{2x^2-7x+3}{\operatorname{tg}(2x-1)}$; **7)** $\lim\limits_{x\to \frac{\pi}{3}}\dfrac{\sin\left(x-\dfrac{\pi}{3}\right)}{1-2\cos x}$; **8)** $\lim\limits_{x\to\infty}\left(\dfrac{x}{2+x}\right)^{3x}$;

9) $\lim\limits_{x\to 0}\left(\dfrac{1-3x}{1-2x}\right)^{1/x}$; **10)** $\lim\limits_{x\to\infty}\left(x\left(\ln(2+x)-\ln x\right)\right)$; **11)** $\lim\limits_{x\to 1}(4-3x)^{5x^2/(1-x)}$;

12) $\lim\limits_{x\to 0}(\cos 2x)^{1/\sin^2 x}$; **13)** $\lim\limits_{x\to 0}\dfrac{2^{3x}-1}{x}$; **14)** $\lim\limits_{x\to 1}\dfrac{e^x-e}{x-1}$; **15)** $\lim\limits_{x\to 0}\dfrac{\ln(1+7x)}{\sin x}$.

Answers: 16.1 1) 5; **2)** 1/3; **3)** 7/2; **4)** 2/3; **5)** 5/2; **6)** 1/2; **7)** 6; **8)** 9/25; **9)** 4; **10)** 1/2; **11)** 12; **12)** – 5/2; **13)** – 1/6; **14)** $\dfrac{3}{10\sqrt{2}}$; **15)** $-\dfrac{\sqrt{2}}{2}$; **16)** $-\dfrac{\sqrt{2}}{4}$; **17)** 1/3; **18)** 0; **19)** 0; **20)** 2; **21)** 1.

16.21) e^2; **2)** e^2; **3)** e^{-11}; **4)** e^{-2}; **5)** e^{-21}; **6)** $e^{-2/3}$; **7)** 0; **8)** 2; **9)** e^{-1}; **10)** e^{-1}; **11)** 1; **12)** e;

13) e^{2}; **14)** e^{3}; **15)** e^{-4}; **16)** 7; **17)** e^{4}.

16.31) $e^{-1/2}$; **2)** $\sqrt{e}$; **3)** e; **4)** 2/3; **5)** 2; **6)** $1/\ln 3$; **7)** $2\ln a$.

16.41) 4/3; **2)** 8/9; **3)** -8; **4)** 1; **5)** ∞; **6)** $-\dfrac{5}{2}$; **7)** $\dfrac{\sqrt{3}}{3}$; **8)** e^{-6}; **9)** e^{-1}; **10)** 2; **11)** e^{15}; **12)** e^{-2}; **13)** $3\ln 2$; **14)** e; **15)** 7.

Lesson 17. Compare infinitesimal functions. Continuity of functions. Break points

Classroom assignments

17.1 Calculate the limits using the relation theorem of two infinitesimal functions:

1) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 2x}{1-\cos x}$; **2)** $\lim\limits_{x\to 0}\dfrac{\ln(1-x)}{2\,\mathrm{tg}\,3x}$; **3)** $\lim\limits_{x\to 0}\dfrac{\arcsin\frac{x}{\sqrt{1-x^2}}}{\ln(1-x)}$; **4)** $\lim\limits_{x\to 0}\dfrac{e^{5x}-1}{\sin 10x}$;

5) $\lim\limits_{x\to 2}\dfrac{\sin 3(x-2)}{x^2-3x+2}$; **6)** $\lim\limits_{x\to -5}\dfrac{\mathrm{tg}(x+5)}{x^2-25}$; **7)** $\lim\limits_{x\to 0}\dfrac{\sin^4 4x}{\mathrm{arctg}^3\,2x}$; **8)** $\lim\limits_{x\to 0}\dfrac{\ln^2\left(1+\sqrt{7x}\right)}{1-e^{x/3}}$.

17.2 Investigate functions on continuity, establish the nature of break points:

1) $f(x)=\dfrac{x}{x-1}$; **2)** $f(x)=\dfrac{\sin(x-2)}{x-2}$; **3)** $f(x)=3^{\frac{x}{4-x^2}}$; **4)** $f(x)=\dfrac{x^2-2x+1}{x^3-x^2-x+1}$; **5)** $f(x)=\mathrm{arctg}\dfrac{1}{x-3}$;

6) $f(x)=\dfrac{|x+1|}{x+1}$; **7)** $f(x)=\begin{cases}2^x, & -\infty<x\le 1,\\ x^2+1, & x>1.\end{cases}$; **8)** $f(x)=\begin{cases}\sin x, & -\infty<x\le 1,\\ x^2-3, & 1<x<2,\\ x-1, & x\ge 2.\end{cases}$ **9)** $f(x)=\dfrac{5^{\frac{1}{x-2}}-1}{5^{\frac{1}{x-2}}+1}$;

10) $f(x)=\dfrac{x^3+1}{x+1}$.

Homework

17.3 Calculate the next limits:

1) $\lim\limits_{x\to 0}\frac{\ln(1+7x)}{\sin 2x}$; **2)** $\lim\limits_{x\to 0}\frac{e^{\sin 7x}-1}{x^2+3x}$; **3)** $\lim\limits_{x\to\frac{1}{2}}\frac{4x^2-1}{\arcsin(1-2x)}$; **4)** $\lim\limits_{x\to 2}\frac{x^2-4}{\operatorname{tg}(x^2-3x+2)}$;

5) $\lim\limits_{x\to 0}\frac{\sqrt{1+\sin 3x}-1}{\ln(1+\operatorname{tg}2x)}$; **6)** $\lim\limits_{x\to 3}\frac{\arcsin^3(x-3)}{\left(e^{x-3}-1\right)^2\operatorname{arctg}x}$.

17.4 Investigate the continuity of the function given below; establish the nature of its break points:

1) $f(x)=\frac{\operatorname{tg}x}{x^2+2x}$; **2)** $f(x)=\frac{1}{1+3^{\frac{1}{x}}}$; **3)** $f(x)=\begin{cases}\sqrt{4-x^2}, -2\le x\le 2,\\ x-2, \quad 2<x\le 4,\\ -2\sqrt{x}, \quad x>4.\end{cases}$ Plot the graphs for the functions;

4) $f(x)=\frac{e^x-e^{-x}}{x}$; **5)** $f(x)=\begin{cases}x^3+1, \quad -\infty\le x\le 0,\\ \cos x, \quad 0<x\le\pi,\\ x-\pi-1, \; x>\pi.\end{cases}$ Plot the graphs for the functions;

6) $f(x)=\frac{1-\cos x}{2x^2-x^3}$.

Answers: 17.1 1) 3; **2)** $-\frac{1}{6}$; **3)** – 1; **4)** $\frac{1}{2}$; **5)** 3; **6)** $-\frac{1}{10}$; **7)** 0; **8)** – 21.

17.21) $x=1$ – break point of the second kind; **2)** $x=2$ – point of removable break, $f(2)=1$; **3)** $x=\pm 2$ – breaks point of the second kind; **4)** $x=1$ – point of removable break, $f(1)=\frac{1}{2}$; $x=-1$ – break point of the second kind; **5)** $x=3$ – first kind of breaking point; **6)** $x=-1$ –first kind of breaking point; **7)** the function is continuous at $x\in R$; **8)** $x=1$ – first kind of breaking point; **9)** $x=2$

– first kind of breaking point; **10)** $x=-1$ – point of removable break; $f(-1)=3$.

17.31) 7/2; **2)** 7/3; **3)** – 2; **4)** 4; **5)** 3/4; **6)** 0.

17.41) $x=0$ – point of removable break, $f(0)=\frac{1}{2}$; $x=-2,\ x=\frac{\pi}{2}+\pi k$ $(k=0;\pm1;\pm2$ etc.) – break point of the second kind; **2)** $x=0$ – first kind of breaking point; **3)** $x=4$– first kind of breaking point; **4)** $x=0$ – point of removable break, $f(0)=2$; **5)** everywhere continuous; **6)** $x=0$ – point of removable break, $f(0)=\frac{1}{4}$; $x=2$ – break point of the second kind**.**

Lesson 18. Derivative of a function, its geometric and physical meaning

Classroom assignments

18.1 Find the derivations for the next functions:

1) $y=7x^2$; **2)** $y=\sqrt{x}$; **3)** $y=5(\operatorname{tg}x-x)$.

18.2 Find derived functions:

1) $y=5x^4-8\sqrt[7]{x^3}+\frac{7}{x^5}+4$; **2)** $y=x^3\sin 2x$; **3)** $y=\frac{x^4+2x+3}{x^5-1}$; **4)** $y=(x^5+3x-7)^4$;

5) $y=\sqrt[3]{\left(\frac{x^3+1}{x^7-2}\right)^2}$; **6)** $y=\sqrt{x}\cdot\ln(2x^3+3x^2-2)$; **7)** $y=\frac{\sin 4x-\cos\frac{x}{2}}{\sin 3x+\cos x}$; **8)** $y=e^{-x^2}\cdot\log_3\frac{1}{x}$;

9) $y=\sqrt{x^3}\cdot\arccos\frac{x}{2}-\sqrt{4-x^2}$; **10)** $y=-\operatorname{ctg}^3\frac{x}{4}-2\ln\sin\frac{x}{2}$; **11)** $y=\operatorname{arctg}(\sqrt{x}+2)$;

12) $y=\frac{2}{2x-1}-\frac{1}{x}$; **13)** $y=\cos^5\left(\sin\frac{2x}{3}\right)+\sin\left(\cos\frac{x}{4}\right)$; **14)** $y=2^{-\sqrt{x}/\ln x}$;

15) $y = \arcsin\frac{2}{x} + \arccos\frac{1}{2}$; **16)** $y = \log_a \ln x$.

18.3 Solve the equation $f'(x) - \frac{2}{x} f(x) = 0$, if $f(x) = x^3 \ln x$.

18.4 Solve the next inequalitiy: $f'(x) + \varphi'(x) \geq 0$ where $f(x) = 2x^3 + 12x^2$, $\varphi(x) = 9x^2 + 72x$.

18.5 Calculate the values of the derivatives of the specified functions at the specified values of the independent variable:

1) $f(x) = \sqrt{x^2+3} + \frac{2x}{x+1} + 6;\ f'(1) = ?$ **2)** $f(x) = \frac{x}{3} - \frac{3}{x} + \sin\frac{\pi}{6};\ f'(3) = ?$

3) $f(x) = \sin 8x \cos 4x;\ f'\left(\frac{\pi}{3}\right) = ?$ **4)** $f(x) = \frac{2\cos x}{1+\sin x};\ f'\left(\frac{\pi}{2}\right) = ?$

5) $f(x) = \frac{\operatorname{arctg} x}{\sqrt{x^3}};\ f'(1) = ?$ **6)** $f(x) = 4e^{-x^2} \cdot \arcsin\frac{x}{2};\ f'(0) = ?$

7) $f(x) = 3^{-\sqrt{2x}};\ f'(2) = ?$ **8)** $f(x) = \ln\frac{2x}{1+4x};\ f'(2) = ?$

9) $f(x) = 5(x^2 - x) \cdot \cos^2 x;\ f'(0) = ?$ **10)** $f(x) = \frac{1}{2}\sqrt{x^2 - 1 + \sqrt[3]{x}};\ f'(1) = ?$

11) $f(x) = \frac{2x-4}{\sin^2 x};\ f'\left(\frac{\pi}{2}\right) = ?$

18.6 Write the equations of tangent and normal to the graph of the function $y = x^2 + 4$ at the point $M(1;5)$.

18.7 Write the equations of tangent and normal to the graph of the function $y = \operatorname{tg} x$ at the point with abscissa $x_0 = \frac{4\pi}{3}$.

18.8 A body with a mass of 7 moves in a straight line according to the law $y = t^2 + t + 4$. Determine the kinetic energy of the body at a point in time $t = 3$.

18.9 The radius of the ball changes at a speed of 6 cm / s. At what speed the volume and surface of the ball change?

18.10 Find the current strength in the conductor, if the charge passing through the cross-section of the conductor changes according to the law $q = 2t + e^{-3t}$ (Cl).

18.11 A material point moves along a circle so that the angular displacement φ changes according to the law $\varphi = 6{,}5 + 7t + 3{,}5t^2 + 2t^3$ (rad). Find the angular velocity of the material point to the point at time $t = 2s$ from the beginning of the motion.

Homework

18.12 Find derived functions:

1) $y = 5e^{2x} \cdot \sqrt{x}$; **2)** $y = \frac{\cos 3x}{\operatorname{tg}\sqrt{x}}$; **3)** $y = x^2 \arcsin 8x$; **4)** $y = \frac{x^3 \cdot \operatorname{ctg} 2x}{\sqrt{x+1}}$;

5) $y = 3^{\arcsin x/2}$; **6)** $y = 5\log_2 \sin\frac{x}{7}$; **7)** $y = \frac{3}{2x-7} - \frac{4}{x}$; **8)** $y = \ln^3\left(8x - 4^{-x^2}\right)$;

9) $y = \arccos\frac{1}{\sqrt{x}}$; **10)** $y = \operatorname{arctg}\sqrt{x\sqrt{x\sqrt{x}}}$; **11)** $y = \frac{\sqrt{x-x^2}}{e^{-x^3}}$; **12)** $y = \frac{a}{\sqrt[3]{x^2}} + \frac{\sqrt[5]{x^3}}{b}$.

18.13 Make up the equations of tangent and normal to the graph of the function $y = e^{1-x^2}$ at the point with abscissa $x_0 = -1$.

18.14 Calculate the values of the derivatives of the specified functions at the specified values of the independent variable:

1) $f(x) = \sqrt{x^2+5} + \frac{14x}{2x-1};\ f'(2) = ?;$ **2)** $f(x) = \frac{2x}{3} - \frac{3}{5x} + \sin 9x;\ f'(\pi) = ?;$

3) $f(x) = \frac{1}{6}\sin 6x \cdot \cos 3x;\ f'\left(\frac{\pi}{3}\right) = ?;$ **4)** $f(x) = \frac{\mathrm{tg}\,x}{\sqrt{x}}x;\ f'\left(\frac{\pi}{4}\right) = ?;$

5) $f(x) = \ln\frac{2x^2}{1+4\sqrt{x}};\ f'(2) = ?;$ **6)** $f(x) = \arccos\sqrt{1-2x} + \sqrt{2x-4x^2};\ f'\left(\frac{1}{2}\right) = ?;$

7) $f(x) = \mathrm{arctg}\,x\frac{1+x}{1-x};\ f'(0) = ?$

Answers: 18.3 $x = \frac{1}{e}$. **18.4** $x \in (-\infty;-4] \cup [-3;+\infty)$**.**

18.5 1) $f'(1) = 7$; **2)** $f'(3) = \frac{2}{3}$; **3)** $f'\left(\frac{\pi}{3}\right) = 5$ **4)** $f'\left(\frac{\pi}{2}\right) = -1$; **5)** $f'(1) = \frac{1}{2} - \frac{3\pi}{8}$;

6) $f'(0) = 2$; **7)** $f'(2) = -\frac{\ln 3}{18}$; **8)** $f'(2) = \frac{1}{18}$; **9)** $f'(0) = -5$; **10)** $f'(1) = \frac{7}{12}$; **11)** $f'\left(\frac{\pi}{2}\right) = 2$.

18.6 $y = 2x+3;\ x+2y-11=0$. **18.7** $y = 3x-\pi$. **18.8** $K = \frac{49}{2}$. **18.9** $v' = 24\pi R^2,\ s' = 48\pi R$.

18.10 $I = 2-3e^{-3t}$. **18.11** $\omega = 45$ pae/c. **18.13** $2x-y+3=0;\ x+2y-1=0$.

18.14 **1)** $f'(2)=-\frac{8}{9}$; **2)** $f'(\pi)=\frac{9-125\pi^2}{15\pi^2}$; **3)** $f'\left(\frac{\pi}{3}\right)=-6$; **4)** $f'\left(\frac{\pi}{4}\right)=\frac{4\pi-1}{\sqrt{\pi^3}}$;

5) $f'(2)=\frac{3\sqrt{2}+1}{4\sqrt{2}+1}$; **6)** $f'\left(\frac{1}{2}\right)=0$; **7)** $f'(0)=-1$.

Lesson 19. A derivative of a function. Logarithmic derivative

Classroom assignments

19.1 Find derived functions:

1) $y=\sqrt{x+2\sqrt{x+3\sqrt{x}}}+\operatorname{sh}x$; **2)** $y=\log_x e$; **3)** $y=\frac{1}{\cos^n(m+1)x}$; **4)** $y=\log_2 \ln^n mx$;

5) $y=e^{-2x}\operatorname{ch}5x$; **6)** $y=\operatorname{arcctg}(\operatorname{th}x)$; **7)** $y=\frac{\operatorname{ctg}4x}{\operatorname{cth}3x}$; **8)** $y=5\operatorname{sh}\frac{1}{x}$.

19.2 Using pre-logarithm, to find derived functions:

1) $y=\frac{(x-3)^2(2x-1)^5}{(4x+1)^3}$; **2)** $y=\sqrt[3]{\frac{(4x-7)^2(12x-x^2)^8}{(2-3x)^5}}$; **3)** $y=(2x)^{\sin x/3}$;

4) $y=(\arcsin 3x)^{\sqrt{x}}$; **5)** $y=(\operatorname{tg}8x)^{x^9}$; **6)** $y=x^{x^3}$; **7)** $y=\left(\cos\frac{1}{x}\right)^{\arcsin 5x}$;

8) $y=(\log_2 x)^{5/x}$; **9)** $y=\operatorname{arctg}2x\cdot(1+4x)^{\sqrt{x}}$; **10)** $y=(\operatorname{th}6x)^{e^{-x^2}}$.

Homework

19.3 Find derived functions:

1) $y = a^{\sqrt{\cos x}} \cdot \sqrt[a]{\cos x}$; **2)** $y = \arcsin\left(\frac{1}{\operatorname{ch} x}\right)$; **3)** $y = x^{-\operatorname{tg} 6x}$; **4)** $y = (\operatorname{acctg} 2x)^{\ln 3x}$; **5)** $y = x^{x^x}$;

6) $y = \frac{(3x+1)^4 \sqrt[5]{2-x}}{\sqrt[5]{(3-x)^4} \cdot x^{4/3}}$; **7)** $y = x^3 \cdot \sqrt{\frac{1-x}{\sqrt[5]{3x-8}}}$; **8)** $y = x^{4^x}$; **9)** $y = \left(\frac{1}{\sqrt{x}}\right)^{1/\sqrt{x}}$; **10)** $y = (\log_x 7)^x$.

19.4 Calculate the values of the derivatives of the specified functions at the specified values of the independent variable:

1) $f(x) = (3x)^{x^4}$; $f'(1) = ?$ **2)** $f(x) = \left(\frac{1}{x^2}\right)^{\cos x}$; $f'(1) = ?$ **3)** $f(x) = (\cos x)^{1/x}$; $f'(2\pi) = ?$

4) $f(x) = x^{\ln 3x}$; $f'(1) = ?$ **5)** $f(x) = (\sin x)^{\operatorname{tg} 2x}$; $f'\left(\frac{\pi}{2}\right) = ?$ **6)** $f(x) = (\arcsin x)^{2x}$; $f'\left(\frac{1}{2}\right) = ?$

7) $f(x) = (x+1)^{2/x}$; $f'(2) = ?$ **8)** $y = \frac{(x-2)^2 \sqrt{x+1}}{(x-5)^3}$; $f'(1) = ?$ **9)** $y = \frac{(3x-2)^4 \sqrt{5x+1}}{(7x-5)^3}$; $f'(0) = ?$

Answers: **19.4** **1)** $f'(1) = 12\ln 3 + 3$; **2)** $f'(1) = -2\cos 1$; **3)** $f'(2\pi) = 0$; **4)** $f'(1) = \ln 3$;

5) $f'\left(\frac{\pi}{2}\right) = 0$; **6)** $f'\left(\frac{1}{2}\right) = \frac{\pi \ln \pi}{3} - \frac{\pi \ln 6}{3} + \frac{2\sqrt{3}}{3}$; **7)** $f'(2) = 1 - \frac{3}{2}\ln 3$; **8)** $f'(1) = \frac{\sqrt{2}}{64}$;

9) $f'(0) = -\frac{56}{625}$.

Lesson 20. Differentiation of functions specified parametrically and implicitly. Function differential

Classroom assignments

20.1 To find derived functions specified parametrically:

1) $x=t^2+2,\ y=\frac{1}{3}t^3-1;$ **2)** $x=\dfrac{1}{t+1},\ y=\left(\dfrac{t}{t+1}\right)^2;$ **3)** $x=a(\varphi-\sin\varphi),\ y=a(1-\cos\varphi)$; **4)** $x=\ln t,\ y=t^2-1;$ **5)** $x=\arccos\sqrt{t},\ y=\sqrt{t-t^2}$;

6) $x=\operatorname{arctg}t,\ y=\ln(1+t^2)$; **7)** $x=a\cos^3 t,\ y=a\sin^3 t$; **8)** $x=\operatorname{tg}t,\ y=\sin 2t+2\cos 2t$.

20.2 Find the derivative y'_x at the specified point:

1) $x=e^t\cos t,\ y=e^t\sin t;\ t=\dfrac{\pi}{6}$. **2)** $x=\dfrac{3at}{1+t^2},\ y=\dfrac{3at^2}{1+t^2};\ t=2$.

20.3 Find derived functions specified implicitly:

1) $e^x+2x^2y^2-e^y=0$; **2)** $2y\ln y=x$; **3)** $x-y=\arcsin x-\arcsin y$; **4)** $2^x+2^y=2^{x+y}$;

5) $\operatorname{arctg}y=y-x^2$; **6)** $\sin(xy)+\cos(xy)=0$; **7)** $x^{2/3}+y^{2/3}=a^{2/3}$; **8)** $e^x\sin y-e^y\cos x=0$.

20.4 Find y'_x at the point $x=1$, if $x^3-2x^2y^2+5x+y-5=0,\ y(1)=1$.

20.5 Find y'_x at the point $(0,1)$, if $e^y+xy=e$.

20.6 Find the next function differentials:

1) $y=x\operatorname{tg}^3 x$; **2)** $y=\sqrt{\operatorname{arctg}x}+(\arcsin x)^2$; **3)** $y=\ln(x+\sqrt{4+x^2})$; **4)** $y^5+y-x^2=1$.

20.7 Find the approximate value of the function $y(x)=e^{x^2-x}$ at $x=1,2$.

20.8 Calculate approximately:

1) $\arcsin 0{,}05$; **2)** $\ln 1{,}2$; **3)** $\sqrt[4]{17}$; **4)** $\operatorname{tg}44^\circ 56'$.

Homework

20.9 Find the derivative y'_x:

1) $x=\frac{t+1}{t}, y=\frac{t-1}{t}$; **2)** $x=e^t\sin t,\ y=e^t\cos t$.

20.10 Make sure that the function specified parametrically by the equations $x=\frac{1+\ln t}{t^2},\ y=\frac{3+2\ln t}{t}$, satisfies the ratio $yy'=2x(y')^2+1$.

20.11 Find the derivatives for the functions specified implicitly:

1) $x^3+y^3-3axy=0$; **2)** $\sin(xy)+\cos(xy)=\mathrm{tg}(x+y)$.

20.12 Make sure that the function *y* defined by the equation $xy-\ln y=1$, satisfies the ratio $y^2+(xy-1)\cdot y'=0$.

20.13 Find the function differentials:

1) $y=x\arcsin x+\sqrt{1-x^2}-3$; **2)** $e^y=x+y$.

20.14 Calculate approximately: **1)** $\sin 29^\circ$; **2)** $\sqrt{\frac{(2{,}037)^2-3}{(2{,}037)^2+5}}$.

20.15 How much approximately will the area of the radius circle change? $R=3$ cm, if the radius will increase by 0,1 cm?

Answers: **20.1** **1)** $\frac{t}{2}$; **2)** $-\frac{2t}{t+1}$; **3)** $\frac{\sin\varphi}{1-\cos\varphi}=\mathrm{ctg}\frac{\varphi}{2}$; **4)** $2t^2$; **5)** $2t-1$; **6)** $2t$; **7)** $-\mathrm{tg}t$;

8) $2(\cos 2t-2\sin 2t)\cos^2 t$.

20.2 **1)** $\frac{1}{2}(\sqrt{3}+1)^2$; **2)** $-\frac{4}{3}$. **20.3** **1)** $\frac{e^x+4xy^2}{e^y-4x^2y}$; **2)** $\frac{1}{2(\ln y+1)}$; **3)** $\frac{\left(\sqrt{1-x^2}-1\right)\sqrt{1-y^2}}{\left(\sqrt{1-y^2}-1\right)\sqrt{1-x^2}}$;

4) $\frac{2^x-2^{x+y}}{2^{x+y}-2^y}$; **5)** $\frac{2x\left(1+y^2\right)}{y^2}$; **6)** $-\frac{y}{x}$; **7)** $-\sqrt[3]{\frac{y}{x}}$; **8)** $\frac{e^y\sin x+e^x\sin y}{e^y\cos x-e^x\cos y}$.

20.4 $\frac{4}{3}$**.** **20.5** $-e^{-1}$**.** **20.6.** **1)** $\mathrm{tg}^2 x\left(\mathrm{tg}x+\frac{3x}{\cos^2 x}\right)dx$**;** **2)** $\left(\frac{1}{2\sqrt{\mathrm{arctg}x}}\cdot\frac{1}{1+x^2}+\frac{2\arcsin x}{\sqrt{1-x^2}}\right)dx$**;**

3) $\frac{dx}{\sqrt{4+x^2}}$ **; 4)** $\frac{2xdx}{5y^4+1}$**. 20.7** 1,2**. 20.8** **1)** 0,05**; 3)** 0,2; **4)** 2,02. **20.9** **1)** –1; 2) $\frac{1-\mathrm{tg}t}{1+\mathrm{tg}t}$.

20.11 **1)** $\frac{ay-x^2}{y^2-ax}$; **2)** $-\frac{y\cos^2(x+y)(\cos(xy)-\sin(xy))-1}{x\cos^2(x+y)(\cos(xy)-\sin(xy))-1}$. **20.13** **1)** $\arcsin x dx$; **2)** $\frac{dx}{e^y-1}$.

20.14 **1)** 0,485; **2)** 0,355. **20.15** 0,6π.

Lesson 21. Derivatives and differentials of higher orders

Classroom assignments

21.1 Find the second order derivatives from the following functions:

1) $y=\cos^2 x$; **2)** $y=\mathrm{arctg}x^2$; **3)** $y=\log_2\sqrt[3]{1-x^2}$;

4) $y=\frac{1}{3}x^2\sqrt{1-x^2}+\frac{2}{3}\sqrt{1-x^2}+x\arcsin x$; **5)** $y=\left(1+x^2\right)\mathrm{arctg}x$; **6)** $y=e^{\sqrt{x}}$.

21.2 Show that the function $y=c_1e^{2x}+c_2e^{3x}$ for any constants c_1 and c_2 satisfies the equation $y''-5y'+6y=0$.

21.3 Find the second order derivatives of functions specified implicitly:

1) $y=1+xe^y$; **2)** $x^3+y^3=3xy$; **3)** $\mathrm{arctg}y=y-x$; **4)** $y=x+\ln y$;

5) $x+y=e^{x-y}$; **6)** $y=\sin(x+y)$.

21.4 Find the second order derivatives of functions specified parametrically:

1) $x=t^2+2, y=\frac{1}{3}t^3-1$; **2)** $x=\arcsin t, y=\sqrt{1-t^2}$;

3) $x = a\cos^2 t, y = a\sin^2 t$; **4)** $x = \ln t, y = t^2 - 1$;

5) $x = a(\varphi - \sin\varphi),\ y = a(1 - \cos\varphi)$; **6)** $x = 1 + e^{\alpha t},\ y = \alpha t + e^{-\alpha t}$.

21.5 Find the differentials of the first, second and 3rd orders of the function $y = (2x-3)^3$.

21.6 Find the second order function differentials:

1) $y = e^{-x^2}$; **2)** $xy + y^2 = 1$.

21.7 Find the 3rd order differential of the function $y = \dfrac{\ln x}{x}$.

21.8 Find approximate value $\sqrt[5]{31}$ to two decimal places.

Homework

21.9 Find derivatives of the second order for the following functions:

1) $y = \sqrt{1-x^2}\ \arcsin x$; **2)** $y = \ln\left(x + \sqrt{1+x^2}\right)$;

3) $y = -\dfrac{1}{9}x\sin 3x - \dfrac{2}{27}\cos 3x$; **4)** $y = \dfrac{1}{1+x^3}$.

21.10 Find $y^{(n)}(x)$ for the function $y = e^{-x}$.

21.11 Find $\dfrac{d^2y}{dx^2}$ for the next functions:

1) $e^{x+y} = xy$; **2)** $x = \dfrac{1}{\cos t}, y = \operatorname{tg} t$;

3) $x^2 + y^2 + xy - 4 = 0$; **4)** $x = \operatorname{arctg} t,\ y = \ln\left(1+t^2\right)$.

21.12 Calculate the value of the second-order derivative of the function y given by the equation $x^2+2y^2-xy+x+y=4$ at the point $M(1;1)$.

21.13 Prove that the function $y=e^{4x}+2e^{-x}$ satisfies the equation $y'''-13y'-12y=0$. Write for this function d^3y.

21.14 Calculate approximate function $y=\sqrt[3]{x^2-5x+12}$ for the value $x=1{,}3$ at two decimal places.

Answers: **21.1** **1)** $-2\cos 2x$; **2)** $\frac{2-6x^4}{(1+x^4)^2}$; **3)** $-\frac{2}{3\ln 2}\cdot\frac{x^2+1}{(x^2-1)^2}$; **4)** $2\sqrt{1-x^2}$;

5) $\frac{2x}{1+x^2}+2\operatorname{arctg}x$; **6)** $\frac{e^{\sqrt{x}}(\sqrt{x}-1)}{4x\sqrt{x}}$. **21.3** **1)** $\frac{(3-y)e^{2y}}{(2-y)^3}$; **2)** $\frac{-2xy}{(y^2-x)^3}$; **3)** $\frac{-2(1+y^2)}{y^5}$;

4) $-\frac{y}{(y-1)^3}$; **5)** $\frac{4(x+y)}{(x+y+1)^3}$; **6)** $-\frac{y}{(1-\cos(x+y))^3}$. **21.4** **1)** $\frac{t^2+1}{4t^3}$; **2)** $-\sqrt{1-t^2}$; **3)** 0; **4)** $4t^2$;

5) $-\frac{1}{a(1-\cos\varphi)^2}$; **6)** $2e^{-3\alpha t}-e^{-2\alpha t}$. **21.5** $6(2x-3)^2dx$; $24(2x-3)dx^2$; $48dx^3$.

21.6 **1)** $e^{-x^2}(4x^2-2)dx^2$; **2)** $\frac{2dx^2}{(x+2y)^3}$. **21.7** $\frac{2\ln x-3}{x^3}dx^3$. **21.8** 1,99**.**

21.9 **1)** $-\frac{\arcsin x+x\sqrt{1-x^2}}{\sqrt{(1-x^2)^3}}$; **2)** $-\frac{x}{\sqrt{(1+x^2)^3}}$; **3)** $x\sin 3x$; **4)** $\frac{6x(2x^3-1)}{(x^3+1)^3}$. **21.10** $(-1)^n e^{-x}$.

21.11 **1)** $-\frac{y((x-1)^2+(y-1)^2)}{x^2(y-1)^3}$; **2)** $-\operatorname{ctg}^3 t$; **3)** $-\frac{24}{(x+2y)^3}$; **4)** $2(1+t^2)$. **21.12** – 1.

21.13 $(64e^{4x}-2e^{-x})dx^3$. **21.14 1**,93.

Lesson 22. Lopital-Bernoulli rule

Classroom assignments

22.1 Calculate the next limits:

1) $\lim\limits_{x\to 0}\dfrac{\sin 2x-\sin 3x}{\arcsin x}$; **2)** $\lim\limits_{x\to \infty}\dfrac{x^3}{3^x}$; **3)** $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$; **4)** $\lim\limits_{x\to 1}\dfrac{x^3-4x^2+5x-2}{x^3-5x^2+7x-3}$;

5) $\lim\limits_{x\to 0}\dfrac{x^4}{x^2+2\cos x-2}$; **6)** $\lim\limits_{x\to 1}\dfrac{2^{\ln x}-x}{\ln x}$; **7)** $\lim\limits_{x\to a+0}\dfrac{\ln(x-a)\cdot\cos x}{\ln\left(e^x-e^a\right)}$; **8)** $\lim\limits_{x\to 1}\dfrac{5x^2+x-1}{x^2+4x+2}$;

9) $\lim\limits_{x\to +\infty}\dfrac{\pi-2\operatorname{arctg} x}{e^{3/x}-1}$; **10)** $\lim\limits_{x\to 0}\dfrac{\ln x}{1+2\ln\sin x}$; **11)** $\lim\limits_{x\to 0}\dfrac{\operatorname{tg} x-x}{x-\sin x}$; **12)** $\lim\limits_{x\to +0}\dfrac{\operatorname{ctg} x}{\ln 2x}$;

13) $\lim\limits_{x\to \infty} x\cdot\sin\dfrac{3}{x}$; **14)** $\lim\limits_{x\to 0} x^2\cdot e^{1/x^2}$; **15)** $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$; **16)** $\lim\limits_{x\to 0}\left(\operatorname{ctg} x-\dfrac{1}{x}\right)$;

17) $\lim\limits_{x\to \pi}(\pi-x)\cdot\operatorname{tg}\dfrac{x}{2}$; **18)** $\lim\limits_{x\to \pi/2}\left(\dfrac{x}{\operatorname{ctg} x}-\dfrac{\pi}{2\cos x}\right)$; **19)** $\lim\limits_{x\to 1}\sin(x-1)\cdot\operatorname{tg}\dfrac{\pi x}{2}$;

20) $\lim\limits_{x\to 0}\left(\dfrac{1}{\ln\left(x+\sqrt{1+x^2}\right)}-\dfrac{1}{\ln(1+x)}\right)$; **21)** $\lim\limits_{x\to 0} x^{\sin x}$; **22)** $\lim\limits_{x\to 0} x^{3/(1+\ln x)}$; **23)** $\lim\limits_{x\to \infty}\left(x+10^x\right)^{1/x}$;

24) $\lim\limits_{x\to 0}(\cos 2x)^{1/x^2}$; **25)** $\lim\limits_{x\to 0}\left(e^x+x\right)^{1/x}$; **26)** $\lim\limits_{x\to 0}(\operatorname{ctg} x)^{1/\ln x}$; **27)** $\lim\limits_{x\to \pi/2}(\operatorname{tg} x)^{2x-\pi}$;

28) $\lim\limits_{x\to \infty}\left(1+\dfrac{1}{x^2}\right)^x$; **29)** $\lim\limits_{x\to 0}(\arcsin x)^{\operatorname{tg} x}$; **30)** $\lim\limits_{x\to \pi/2}(\pi-2x)^{\cos x}$.

Homework

22.2 Calculate the next limits:

1) $\lim\limits_{x\to\infty}\dfrac{x+2\ln x}{x}$; **2)** $\lim\limits_{x\to 0}\dfrac{x^3}{x-\sin x}$; **3)** $\lim\limits_{x\to 0}\dfrac{\ln\sin 5x}{\ln\sin 2x}$; **4)** $\lim\limits_{x\to -\infty}\dfrac{3x^3+x+1}{e^{-x}}$; **5)** $\lim\limits_{x\to 0}\dfrac{x-\operatorname{arctg}x}{x^3}$;

6) $\lim\limits_{x\to 0}\dfrac{2^x-3^x}{x\sqrt{1-x^2}}$; **7)** $\lim\limits_{x\to 0}\dfrac{e^{-3x}-e^{\sin x}}{x}$; **8)** $\lim\limits_{x\to 0}\left(\dfrac{1}{\operatorname{arctg}x}-\dfrac{1}{x}\right)$; **9)** $\lim\limits_{x\to 0}\left(\dfrac{1}{2x^2}-\dfrac{1}{2x\operatorname{tg}x}\right)$;

10) $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$; **11)** $\lim\limits_{x\to 0} x\cdot\operatorname{ctg}\pi x$; **12)** $\lim\limits_{x\to 1}\ln x\cdot\ln(x-1)$; **13)** $\lim\limits_{x\to\infty}(\ln x)^{1/x}$;

14) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\operatorname{tg}x}$; **15)** $\lim\limits_{x\to 0} x^{\frac{1}{\ln(e^x-1)}}$; **16)** $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

Answers: 22.1 1) – 1; **2)** 0; **3)** 2; **4)** 1/2; **5)** 12; **6)** $\ln 2-1$; **7)** $\cos a$; **8)** 5; **9)** 2/3; **10)** 1/2; **11)** 2; **12)** ∞; **13)** 3; **14)** + ∞; **15)** 1/2; **16)** 0; **17)** 2; **18)** – 1; **19)** $-2/\pi$; **20)** $-\dfrac{1}{2}$; **21)** 1; **22)** e^3; **23)** 10; **24)** e^{-2}; **25)** e^2; **26)** e^{-1}; **27)** 1; **28)** 1; **29)** 1; **30)** 1.

22.21) 1; **2)** 6; **3)** 1; **4)** 0; **5)** 1/3; **6)** $\ln\dfrac{2}{3}$; **7)** – 4; **8)** 0; **9)** 1/6; **10)** 1/2; **11)** 1/π; **12)** 0; **13)** 1; **14)** 1; **15)** *e*; **16)** $e^{-1/6}$.

Lesson 23. Taylor formula

Classroom assignments

23.1 Decompose the polynomial $x^4 - 2x^2 + 13x + 9$ to the powers of the two-member $x+2$.

23.2 Decompose the polynomial $x^3 + 3x^2 - 2x + 4$ to the powers of the two-member $x+1$.

23.3 For the polynomial $x^4 + 4x^2 - x + 3$ write the second-order Taylor formula at the point $x_0 = 1$. Record the residual term in the Lagrange form.

23.4 Write the 3rd order Taylor formula for the function $f(x) = 10^x$ at the point $x_0 = 0$.

23.5 Write the 3rd order Taylor formula for the function $f(x) = \dfrac{x}{x-1}$ at the point $x_0 = 2$.

23.6 Write the 3rd order Taylor formula for the function $f(x) = \mathrm{tg}\, x$ at trhe point $x_0 = 0$.

23.7 Derive an approximate formula $\sin x \approx x - \dfrac{x^3}{6}$ and evaluate its accuracy for $|x| < 0{,}05$.

23.8 Derive approximate formulas and estimate their error at $|x| < 1$:

1) $\sqrt{1+x} \approx 1 + \dfrac{1}{2}x - \dfrac{1}{8}x^2$; **2)** $\sqrt[3]{1+x} \approx 1 + \dfrac{1}{3}x - \dfrac{1}{9}x^2$.

23.9 Check that when calculating the values of the function e^x with $0 < x \le \dfrac{1}{2}$ an approximate formula $e^x \approx 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$, the permissible error is less than 0.01. Taking advantage of this, to find $e^{0,2}$ with three correct numbers.

23.10 Calculate accurately $\cos 10°$ up to 10^{-4}.

23.11 Calculate accurately up to 10^{-3}:

1) $\sqrt[5]{33}$; **2)** $\ln 1{,}05$.

23.12 Find the next limits using the MacLoren formula decomposition with a residual term at the Peano form:

1) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$; **2)** $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2+x^3}$;

3) $\lim\limits_{x\to 0}\dfrac{xe^{2x}+xe^x-2e^{2x}+2e^x}{\left(e^x-1\right)^3}$; **4)** $\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x\sqrt{1+x}}{x^2}$.

Homework

23.13 Decompose the polynomial $x^4-5x^3+x^2-3x+4$ to the powers of the two-member $x-4$.

23.14 Write a 3rd order Taylor formula for the function $f(x)=\dfrac{1}{\sqrt{x}}$ at the point $x_0=1$.

23.15 Write a 3rd order Taylor formula for the function $y=\arcsin x$ at the point $x_0=0$.

23.16 Write a 3rd order Taylor formula for the function $y=\dfrac{1}{x}$ at the point $x_0=-1$.

23.17 Write a nth-order McLaurin formula for the function $f(x)=xe^x$, $x_0=0$.

23.18 Calculate approximate values with specified precision Δ:

1) $\sin 1°, \Delta=10^{-4}$; **2)** $\sqrt{e}, \Delta=10^{-3}$; **3)** $\sqrt[10]{1027}, \Delta=10^{-4}$; **4)** $\cos 5°, \Delta=10^{-5}$.

23.19 Find the next limits using the MacLoren formula decomposition with a residual term in the Peano form:

1) $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^2\sin x}$; **2)** $\lim\limits_{x\to 0}\dfrac{1-\cos x}{e^x-1-x}$; **3)** $\lim\limits_{x\to 0}\dfrac{x\sin x}{\sqrt[3]{1+3x}-\sqrt{1+2x}}$; **4)** $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$.

Answers: 23.1 $f(x)=-9-11(x+2)+22(x+2)^2-8(x+2)^3+(x+2)^4$;

23.2 $f(x)=(x+1)^3-5(x+1)+8$; **23.3** $f(x)=7+11(x-1)+10(x-1)^2+4(1+\theta(x-1))(x-1)^3$;

23.4 $10^x = 1 + \ln 10 \cdot x + \frac{\ln^2 10}{2!} x^2 + \frac{\ln^3 10}{3!} x^3 + \frac{10^{\theta x} \ln^4 10}{4!} x^4$, $0 < \theta < 1$;

23.5 $\frac{x}{x-1} = 2 - (x-2) + (x-2)^2 - (x-2)^3 + \frac{(x-2)^4}{(1+\theta(x-2))^5}$, $0 < \theta < 1$;

23.6 $\operatorname{tg} x = x + \frac{1 + 2\sin^3 \theta x}{\cos^4 \theta x} \cdot \frac{x^3}{3}$, $0 < \theta < 1$; **23.7** $\Delta < 3 \cdot 10^{-9}$; **23.8** **1)** $\frac{1}{16} \cdot \frac{x^3}{(1+\theta x)^{5/2}}$, $0 < \theta < 1$;

2) $\frac{5}{81} \cdot \frac{x^3}{(1+\theta x)^{8/3}}$, $0 < \theta < 1$; **23.9** 1,221; **23.10** 0,9848; **23.11** **1)** 2,012; **2)** 0,049; **23.12** **1)** 1;

2) 1/2; **3)** 1/6; **4)** – 1; **23.13** $f(x) = -56 + 21(x-4) + 37(x-4)^2 + 11(x-4)^3 + (x-4)^4$;

23.14 $f(x) = 1 - \frac{1}{2}(x-1) + \frac{1 \cdot 3}{2^2 \cdot 2!}(x-1)^2 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}(x-1)^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 4!} \cdot \frac{(x-1)^4}{(1+\theta(x-1))^{9/2}}$, $0 < \theta < 1$;

23.15 $y = x + \frac{x^3}{6} + \frac{x^4}{4!} \cdot \frac{9\theta x + 6\theta^3 x^3}{(1-\theta^2 x^2)^{7/2}}$, $0 < \theta < 1$;

23.16 $y = -1 - (x+1) - (x+1)^2 - \ldots - (x+1)^n + (-1)^{n+1} \frac{(x+1)^{n+1}}{(-1+\theta(x+1))^{n+2}}$, $0 < \theta < 1$;

23.17 $f(x) = x + x^2 + \frac{x^3}{2!} + \ldots + \frac{x^n}{(n-1)!} + \frac{x^{n+1}}{(n+1)!} \cdot (\theta x + n + 1) e^{\theta x}$, $0 < \theta < 1$;

23.18 **1)** 0,0175; **2)** 1,648; **3)** 2,0006; **4)** 0,99619; **23.19** **1)** – 1/6; **2)** 1; **3)** – 2; **4)** 1/2.

Lesson 24. Monotony of functions. Extremum. The highest and lowest values for the functions. Convexity and concavity of function graphs

Classroom assignments

24.1 Find the intervals of increase, decrease and extremum point of functions:

1) $y=15-x^2-2x$; **2)** $y=2x^3-6x^2-18x+7$; **3)** $y=\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x+\frac{9}{4}$;

4) $y=x\sqrt{1-x^2}$; **5)** $y=\frac{2x^2-1}{x^4}$; **6)** $y=x^2e^{-x}$; **7)** $y=\frac{x}{\ln x}$; **8)** $y=\sqrt[3]{x^2-2x}$;

9) $y=\frac{e^x}{x+1}$; **10)** $y=\ln\frac{x+1}{x+2}$.

24.2 Find extreme functions using a second order derivative:

1) $y=4x^3-9x^2+6x$; **2)** $y=e^{3x}-3x+2$; **3)** $y=\frac{2x}{1+x^2}$;

4) $y=x\ln^2 x$; **5)** $y=x^2(a-x)^2$; **6)** $y=x+\sqrt{1-x}$.

24.3 Determine the highest and lowest values of functions in the specified intervals:

1) $y=x^4-2x^2+5, [-2;0]$; **2)** $y=x+2\sqrt{x},\ [0;4]$; **3)** $y=\sqrt[3]{x+1}-\sqrt[3]{x-1},\ [0;1]$;

4) $y=\operatorname{arctg}\frac{1-x}{1+x},\ [0;1]$; **5)** $y=\frac{x^2-1}{x^2+1},\ [-2;1]$; **6)** $y=x\ln x-x;\ [1/e;e]$.

24.4 Find the intervals of convexity, concavity and inflection points of graph functions:

1) $y=\ln\left(1+x^2\right)$; **2)** $y=e^{-8x^2+4x}$; **3)** $y=x^2+\dfrac{1}{x^2}$; **4)** $y=\left(\dfrac{x+1}{x-1}\right)^2$; **5)** $y=x^4-2x^2+3$;

6) $y=\dfrac{3x^4+1}{x^3}$; **7)** $y=\dfrac{2x^2-1}{x^4}$; **8)** $y=x^2\cdot e^{-x}$; **9)** $y=x\sqrt{1-x^2}$; **10)** $y=\sqrt[3]{x^2-2x}$.

24.5 It is required to make a box with a lid, the volume of which would be equal to 72 cm^3, and the sides of the base would be treated as 1: 2. What should be the dimensions of all sides so that the full surface of the box is the smallest?

24.6 Find the ratio between the radius R and the height H of the cylinder having the smallest total surface at a given volume.

24.7 Find the height of the cylinder of the largest volume that can be entered into the ball radius R.

Homework

24.8 Find the intervals of increase, decrease and extremum point for the next functions:

1) $y=\ln x-\mathrm{arctg}x$; **2)** $y=\dfrac{x^3}{x-1}$; **3)** $y=(x-1)e^x$; **4)** $y=x^3-6x^2+16$.

24.9 Find the extremum of the function $y=x+\dfrac{a^2}{x}$, $a>0$ using the second derivative.

24.10 Find the inflection point of the graphs for the next functions:

1) $y=\dfrac{2x-1}{(x-1)^2}$; **2)** $y=x\,\mathrm{arctg}x$; **3)** $y=\dfrac{4x^3}{1-x^3}$; **4)** $y=e^{-x^2/2}$ (Gaussian curve).

24.11 Find the largest and smallest function values in intervals (or in the entire scope of the definition):

1) $y=\dfrac{1-x+x^2}{1+x-x^2}$, $[0;1]$; **2)** $y=xe^{-x^2/2}$; **3)** $y=\ln\left(4-x^2\right)$, $[-1;1]$; **4)** $y=x+2\sqrt{x}$, $[0;4]$.

24.12 From three boards of the same width, a chute is formed to supply water. At what angle α inclination of the side walls to the bottom of the gutter will the cross-sectional area be the greatest?

Answers: 24.1 1) $(-\infty;-1)$ – increases; $(-1;+\infty)$ – decreases; $y_{\max}=y(-1)=16$;

2) $(-\infty;-1)\cup(3;+\infty)$ – increases; $(-1;3)$ – decreases; $y_{\min}=y(3)=-47; y_{\max}=y(-1)=13$;

3) $(1;2)\cup(3;+\infty)$ – increases; $(-\infty;1)\cup(2;3)$ – decreases; $y_{\min}=y(1)=y(3)=0$; $y_{\max}=y(2)=\frac{1}{4}$;

4) $\left(-1;-1/\sqrt{2}\right)\cup\left(1/\sqrt{2};1\right)$ – decreases; $\left(-1/\sqrt{2};1/\sqrt{2}\right)$ – increases; $y_{\min}=y\left(-1/\sqrt{2}\right)=-1/2$; $y_{\max}=y\left(1/\sqrt{2}\right)=1/2$;

5) $(-\infty;-1)\cup(0,1)$ – increases; $(-1;0)\cup(1;+\infty)$ – decreases; $y_{\max}=y(-1)=y(1)=1$;

6) $(0;2)$ – increases; $(-\infty;0)\cup(2;\infty)$ – decreases; $y_{\min}=y(0)=0; y_{\max}=y(2)=4/e^2$;

7) $(e;+\infty)$ – increases; $(0;1)\cup(1;e)$ – decreases; $y_{\min}=y(e)=e$;

8) $(1;+\infty)$ – increases; $(-\infty;1)$ – decreases; $y_{\min}=y(1)=-1$;

9) $(0;+\infty)$ – increases; $(-\infty;-1)\cup(-1;0)$ – decreases; $y_{\min}=y(0)=1$;

10) $(-\infty;-2)\cup(-1;+\infty)$ – increases; $(-2;-1)$ – decreases; there are no extremes.

24.21) $y_{\max}=y\left(\frac{1}{2}\right)=\frac{5}{4}$; $y_{\min}=y(1)=1$; **2)** $y_{\min}=y(0)=3$;

3) $y_{\max}=y(1)=1$; $y_{\min}=y(-1)=-1$; **4)** $y_{\max}=y\left(e^{-2}\right)=4/e^2$; $y_{\min}=y(1)=0$;

5) $y_{\max}=y\left(\frac{a}{2}\right)=\frac{a^4}{16}$; $y_{\min}=y(0)=y(a)=0$; **6)** $y_{\max}=y\left(\frac{3}{4}\right)=\frac{5}{4}$;

24.31) $y_{high.}=y(-2)=13$; $y_{low.}=y(-1)=4$; **2)** $y_{high.}=y(4)=8$; $y_{low.}=y(0)=0$;

3) $y_{high.}=y(0)=2;\ y_{low.}=y(1)=\sqrt[3]{2}$; **4)** $y_{high.}=y(0)=\frac{\pi}{4};\ y_{low.}=y(1)=0$;

5) $y_{high.}=y(-2)=\frac{3}{5};\ y_{low.}=y(0)=-1$; **6)** $y_{high.}=y(e)=0;\ y_{low.}=y(1)=-1$;

24.41) $(-\infty;-1)\cup(1;+\infty)$ – the graph is convex; $(-1;1)$ – the graph is concave; $(-1;\ln 2),(1;\ln 2)$ – there are no inflection points;

2) $\left(0;\frac{1}{2}\right)$ – the graph is convex; $(-\infty;0)\cup\left(\frac{1}{2};+\infty\right)$ – the graph is concave; $(0;1),\left(\frac{1}{2};1\right)$ – there are no inflection points;

3) $(-\infty;0)\cup(0;+\infty)$ – the graph is concave; there are no inflection points;

4) $(-\infty;-2)$ – the graph is convex; $(-2;1)\cup(1;+\infty)$ – the graph is concave; $\left(-2;\frac{1}{9}\right)$ – inflection point;

5) $\left(-\infty;-\frac{\sqrt{3}}{3}\right)\cup\left(\frac{\sqrt{3}}{3};+\infty\right)$ – the graph is concave; $\left(-\frac{\sqrt{3}}{3};\frac{\sqrt{3}}{3}\right)$ – the graph is convex; $\left(-\frac{\sqrt{3}}{3};\frac{22}{9}\right),\left(\frac{\sqrt{3}}{3};\frac{22}{9}\right)$ – inflection point;

6) $(-\infty;0)$ – the graph is convex; $(0;+\infty)$ – the graph is concave; there are no inflection points;

7) $\left(-\infty;-\sqrt{\frac{5}{3}}\right)\cup\left(\sqrt{\frac{5}{3}};+\infty\right)$ – the graph is concave; $\left(-\sqrt{\frac{5}{3}};0\right)\cup\left(0;\sqrt{\frac{5}{3}}\right)$ – the graph is convex; $\left(-\sqrt{\frac{5}{3}};\frac{21}{25}\right)\cup\left(\sqrt{\frac{5}{3}};\frac{21}{25}\right)$ – – inflection point;

8) $\left(-\infty;2-\sqrt{2}\right)\cup\left(2+\sqrt{2};+\infty\right)$ – the graph is concave; $\left(2-\sqrt{2};2+\sqrt{2}\right)$ – the graph is convex; $\left(2+\sqrt{2};\left(2+\sqrt{2}\right)^2e^{-2-\sqrt{2}}\right),\left(2-\sqrt{2};\left(2-\sqrt{2}\right)^2e^{-2+\sqrt{2}}\right)$ – inflection point;

9) $(-1;0)$ – the graph is concave; $(0;1)$ – the graph is convex; $(0;0)$ – inflection point;

10) $(-\infty;0)\cup(2;+\infty)$ – the graph is convex; $(0;2)$ – the graph is concave; $(0;0);\ (2;0)$ – inflection points.

24.5 3,6 and 4 cm; **24.6** *H*=2*R*; **24.7** $H=\frac{2R\sqrt{3}}{3}$; **24.8** **1)** increases throughout the definition area;

2) $(-\infty;1)\cup\left(1;\frac{3}{2}\right)$ – decreases; $\left(\frac{3}{2};+\infty\right)$ – increases; $y_{\min}=y\left(\frac{3}{2}\right)=\frac{27}{4}$;

3) $(-\infty;0)$ – decreases; $(0;+\infty)$ – increases; $y_{\min}=y(0)=-1$;

4) $(-\infty;0)\cup(4;+\infty)$ – increases; $(0;4)$ – decreases; $y_{\max}=y(0)=16$, $y_{\min}=y(4)=-16$;

24.9 $y_{\max}=y(-a)=-2a$; $y_{\min}=y(a)=2a$;

24.10 1) $\left(-\frac{1}{2};-\frac{8}{9}\right)$; **2)** no inflection points; **3)** $(0;0)$, $\left(-\sqrt[3]{\frac{1}{2}};\frac{4}{3}\right)$; **4)** $(-1;e^{-1/2})$, $(1;e^{-1/2})$.

24.11 1) 1 and 3/5; **2)** $1/\sqrt{e}$ and $-1/\sqrt{e}$; **3)** $\ln 4$ and $\ln 3$; **4)** 8 and 0; **24.12** $\alpha=\frac{2\pi}{3}$.

Lesson 25. Asymptotes. Plotting graphs of functions

Classroom assignments

25.1 Find asymptotes the next function graph:

1) $y=\frac{x^2-4x+5}{x-2}$; **2)** $y=\frac{e^x}{x+1}$; **3)** $y=\frac{x^4}{x^3+1}$; **4)** $y=\frac{\ln x}{x}$; **5)** $y=3x+\text{arctg}5x$;

6) $y=\frac{x^2+2}{x^2-9}$; **7)** $y=x+\sin x$; **8)** $y=(x-2)e^{-1/x}$; **9)** $y=\frac{x^2}{\sqrt{x^2-1}}$; **10)** $y=x^2+2/x$.

25.2 Conduct a full study and sketch the graphs for the next functions:

1) $y=\frac{2x^2-1}{x^4}$; **2)** $y=x^2e^{-x}$; **3)** $y=x\sqrt{1-x^2}$; **4)** $y=\sqrt[3]{x^2-2x}$; **5)** $y=x^2\ln x$;

6) $y=\frac{x^2-1}{x^2+1}$; **7)** $y=\frac{x}{(1+x)^3}$; **8)** $y=(x-1)e^x$; **9)** $y=\frac{4x^3}{1-x^3}$; **10)** $y=x^3-6x^2+16$.

Homework

25.3 Find the horizontal and vertical asymptotes of each curve

1) $y=\frac{2-4x^2}{1-4x^2}$; **2)** $y=x\ln\left(e+\frac{1}{x}\right)$; **3)** $y=\frac{x^3}{2(1+x)^2}$; **4)** $y=\frac{2+x^3}{x^2}$.

25.4 Investigate functions and sketch its graphs:

1) $y=\frac{x^3}{1-x^2}$; **2)** $y=xe^{1/x}$; **3)** $y=\ln(1-x^2)$; **4)** $y=e^{2x-x^2}$.

Answers: 25.1 **1)** $x=2;\ y=x-2$; **2)** $x=-1;\ y=0$ (left); **3)** $x=-1;\ y=x$; **4)** $x=0;\ y=0$;

5) $y=3x+\frac{\pi}{2}$ (right), $y=3x-\frac{\pi}{2}$ (left); **6)** $x=3,\ x=-3,\ y=1$; **7)** no;

8) $x=0;\ y=x-3$; **9)** $x=\pm 1;\ y=\pm x$; **10)** $x=0$.

25.31) $x=\pm\frac{1}{2};\ y=1$; **2)** $x=-1/e;\ y=x+\frac{1}{e}$; **3)** $x=-1;\ y=\frac{1}{2}x-1$; **4)** $x=0;\ y=x$.

Lesson 26. Curvature of the curve

Classroom assignments

26.1 Find the curvature and radius of curvature of the line $y=x^2+2$ at the point $(1;3)$.

26.2 Calculate the curvature of the line $y=\sqrt[3]{x}+2$ at any point.

26.3 Calculate the curvature of the ellipse $x^2+9y^2=9$ at its vertices.

26.4 Find the curvature and radius of curvature of the line $xy=4$ at the point $(2;2)$.

26.5 Find the curvature and radius of curvature of the curve $x^2+xy+y^2=3$ at the point $M(1;1)$.

26.6 Calculate the curvature of the curve $r=2(1-\cos\varphi)$ at the point with the given value $\varphi=\pi$

26.7 Calculate the curvature of the curve $x=t^2,\ y=t-\dfrac{t^3}{3}$ at the point $\left(1;\dfrac{2}{3}\right)(t=1)$.

Homework

26.8 Find the curvature and radius of curvature for the line $y=\sqrt{x^3}$ at the point $M(4;8)$.

26.9

Calculate the curvature of the line $y=\dfrac{e^x+e^{-x}}{2}$ at the point with given value x=0.

26.10 Calculate the curvature of the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1$ at the point with given values $x=2,\ y=0$.

26.11 Find the curvature of the curve $x^2+y^2-xy=1$ at the point $M(1;1)$.

26.12 Calculate the curvature of the line $x=\dfrac{t^2}{2},\ y=\dfrac{t^3}{3}$ at the point $M\left(\dfrac{1}{2};\dfrac{1}{3}\right)(t=1)$.

26.13 Calculate the curvature of the line $y=2\cos t,\ y=3\sin t$ at any point.

Answers: 26.1 $K=\frac{2}{5\sqrt{5}}$; $R=\frac{5\sqrt{5}}{2}$ **; 26.2** $K=\frac{1}{6\sqrt[3]{x}}\cdot\sqrt{\left(9x^{4/3}+1\right)^3}$ **; 26.3** $K=3$; $K=\frac{1}{9}$ **;**

26.4 $K=\frac{\sqrt{2}}{4}$ **; 26.5** $K=\frac{1}{3\sqrt{2}}$; $R=3\sqrt{2}$ **; 26.6** $K=\frac{3}{8}$ **; 26.7** $K=\frac{1}{2}$ **;**

26.8 $K=\frac{3}{80\sqrt{10}}$; $R=\frac{80\sqrt{10}}{3}$ **; 26.9** $K=1$**; 26.10** $K=\frac{9}{2}$ **; 26.11** $K=\frac{3}{\sqrt{2}}$ **; 26.12** $K=\frac{1}{2\sqrt{2}}$ **; 26.13** $K=\frac{\left(4\sin^2 t+9\cos^2 t\right)^{3/2}}{6}$.

Раздел контроля знаний

SECTION OF KNOWLEDGE CONTROL

FOR THE EDUCATIONAL DISCIPLINE “MATHEMATICS. PART I”

TYPICAL CALCULATION No. 1

ELEMENTS OF LINEAR ALGEBRA AND ANALYTICAL GEOMETRY

Task 1

Investigate the system of equations and solve it in case of compatibility.

1.1. a) $\begin{cases} x_1 - x_2 - x_3 - x_4 = 1, \\ 2x_1 + x_2 - x_3 + x_4 = 3, \\ 3x_1 + x_4 = 4. \end{cases}$ b) $\begin{cases} 2x_1 - 3x_2 - x_3 = 0, \\ x_1 + x_2 + x_3 = 1, \\ 3x_1 - 2x_2 = 1, \\ x_1 - 2x_2 - 2x_3 = -1. \end{cases}$

1.2. a) $\begin{cases} 2x_1 + x_3 + 2x_4 = 5, \\ x_2 - x_3 + x_4 = 0, \\ 2x_1 + x_2 + 3x_3 = 5. \end{cases}$ b) $\begin{cases} 2x_1 - 3x_2 + x_3 = 0, \\ x_1 + x_2 + x_3 = 1, \\ 4x_1 + 5x_2 - x_3 = -1, \\ 7x_1 + 3x_2 + x_3 = 3. \end{cases}$

1.3. a) $\begin{cases} x_2 + 2x_3 + 3x_4 = 2, \\ x_1 - x_2 - x_3 - 2x_4 = 0, \\ x_1 + x_2 + x_4 = -1. \end{cases}$ b) $\begin{cases} x_1 - x_2 + 2x_3 = 1, \\ 3x_1 + x_2 + x_3 = -2, \\ x_1 + x_2 + x_3 = 3, \\ x_1 - x_2 + x_3 = 0. \end{cases}$

1.4. a) $\begin{cases} 2x_2 + x_3 + 4x_4 = 0, \\ x_1 - x_3 + x_4 = 2, \\ x_1 + 2x_2 + 5x_4 = 1. \end{cases}$ b) $\begin{cases} 3x_1 + 4x_2 - x_3 = 0, \\ x_1 + 2x_2 + x_3 = 0, \\ 2x_1 - x_2 + x_3 = 0. \end{cases}$

1.5. a) $\begin{cases} 4x_2+2x_3-3x_4=0, \\ 3x_1-3x_2+x_4=3, \\ 3x_1+x_2+2x_3-2x_4=3. \end{cases}$

b) $\begin{cases} 3x_1+x_2+x_3=4, \\ x_1+x_3-2x_4=2, \\ 2x_1+x_2+2x_4=2. \end{cases}$

1.6. a) $\begin{cases} x_1-2x_2+3x_3=3, \\ 2x_1+x_3-x_4=1, \\ x_1+2x_2-2x_3-x_4=-2. \end{cases}$

b) $\begin{cases} 2x_1-x_3-2x_4=0, \\ x_1+2x_2-x_3=1, \\ x_2+x_4=2, \\ 3x_1+3x_2-2x_3=0. \end{cases}$

1.7. a) $\begin{cases} 4x_1-2x_3+5x_4=0, \\ 3x_1+x_3-x_4=0, \\ x_1-3x_3+6x_4=0. \end{cases}$

b) $\begin{cases} x_1+x_3-x_4=7, \\ 2x_1+x_2+x_4=6, \\ x_1-x_2+x_3=-5, \\ 4x_1+2x_3=0. \end{cases}$

1.8. a) $\begin{cases} x_1-x_3+x_4=0, \\ 2x_1+x_3-2x_4=0, \\ 3x_1+2x_2-x_4=0. \end{cases}$

b) $\begin{cases} 2x_1+2x_2+x_3=5, \\ x_1-x_3+x_4=0, \\ 3x_1+2x_2+x_4=1, \\ x_2+x_3-x_4=0. \end{cases}$

1.9. a) $\begin{cases} x_1-x_2+x_3+x_4=0, \\ 2x_1+x_2+x_3-x_4=0, \\ x_1+2x_2-2x_4=0. \end{cases}$

b) $\begin{cases} 3x_1-2x_2-x_3=1, \\ x_1+x_2+x_3=0, \\ 5x_2+x_3=7, \\ x_1+3x_2=6. \end{cases}$

1.10. a) $\begin{cases} x_1+x_2-x_4=0, \\ x_2+x_3+x_4=0, \\ x_3-4x_4=0. \end{cases}$

b) $\begin{cases} x_2+x_3-x_4=-2, \\ x_1+x_2-x_3=4, \\ 2x_1+x_2+x_4=3, \\ 3x_1+3x_2=0. \end{cases}$

Task 2

2.1. Calculate $(\vec{a},\vec{b})$, where $\vec{a}=3\vec{m}_1-2\vec{m}_{2;}$ $\vec{b}=\vec{m}_1+4\vec{m}_2;$ $\vec{m}_1,\ \vec{m}_2$ – unit vectors, the angle between which is equal to $\frac{\pi}{4}$.

2.2. Find the projection of the vector $\vec{a}=4\vec{i}-3\vec{j}+4\vec{k}$ onto the direction of the vector $\vec{b}=2\vec{i}+2\vec{j}+\vec{k}$.

2.3. Find $(\vec{a},\vec{b})$, $|\vec{a}|,|\vec{b}|$, if $\vec{a}=2\vec{i}+\vec{j}-\vec{k},\ \vec{b}=\vec{j}+2\vec{k}$.

2.4. The vector $\vec{c}$, collinear to the vector $\vec{a}=5\vec{i}-2\vec{k}$, forms an acute angle with the Oz axis. Find the coordinates of the vector $\vec{c}$, if $|\vec{c}|=3\sqrt{29}$.

2.5. Find $\left(2\vec{a}-3\vec{b},\ \vec{a}-\vec{b}\right)$, if $|\vec{a}|=\sqrt{2}, |\vec{b}|=2$, $\left(\vec{a},\wedge\vec{b}\right)=\frac{\pi}{4}$.

2.6. Find $(\vec{a},\vec{b})$, $|\vec{a}|,|\vec{b}|$, if $\vec{a}=2\vec{m}+3\vec{n}-\vec{p};$ $\vec{b}=\vec{m}-4\vec{p},$ The vectors $\vec{m},\ \vec{n},\ \vec{p}$ forms an orthogonal basis and $|\vec{m}|=2,$ $|\vec{n}|=3,$ $|\vec{p}|=4$.

2.7. Find the length of the vector $\vec{a}=3\vec{m}+4\vec{n}$, if $|\vec{m}|=|\vec{n}|=1,$ $\left(\vec{m}\wedge\vec{n}\right)=\frac{\pi}{3}$.

2.8. Find a vector $\vec{b}$, collinear to the vector $\vec{a}=2\vec{i}+\vec{j}-\vec{k}$ and satisfying the condition $(\vec{a},\vec{b})=3$.

2.9. Find $\left(2\vec{a}-5\vec{b},\ \vec{a}+3\vec{b}\right)$, if $|\vec{a}|=2,$ $|\vec{b}|=3,$ $\left(\vec{a}\wedge\vec{b}\right)=\frac{2\pi}{3}$.

2.10. Calculate the sine of the angle between the diagonals of a parallelogram whose sides are vectors $\vec{a}=2\vec{i}+\vec{j}-\vec{k},\ \vec{b}=\vec{i}-3\vec{j}+\vec{k}$.

Task 3

3.1. Find $\lfloor 2\vec{a}+\vec{b},\ \vec{b}\rfloor$, where $\vec{a}=3\vec{i}-\vec{j}-2\vec{k};$ $\vec{b}=\vec{i}+2\vec{j}-\vec{k}$.

3.2. Calculate the area of a parallelogram built on vectors $\vec{a}=\vec{m}+2\vec{n}$ and $\vec{b}=\vec{m}-3\vec{n}$, if $|\vec{m}|=5;$ $|\vec{n}|=3,$ $\left(\vec{m}\wedge\vec{n}\right)=\frac{\pi}{6}$.

3.3. The vector $\vec{c}$ is perpendicular to the vectors $\vec{a}$ and $\vec{b}$, the angle between $\vec{a}$ and $\vec{b}$ is equal to $\frac{\pi}{6}$. Knowing that $|\vec{a}|=6, |\vec{b}|=3, |\vec{c}|=3$, calculate $(\vec{a},\vec{b},\vec{c})$.

3.4. Find $\left[2\vec{a}-\vec{b},\ 2\vec{a}+\vec{b}\right]$, where $\vec{a}=2\vec{i}-\vec{j}+\vec{k};\ \vec{b}=3\vec{k}-\vec{i}-2\vec{j}$.

3.5. Find the vector $\vec{x}$, if it is known that it is orthogonal to the vectors $\vec{a}=\vec{i}-\vec{j}+3\vec{k},\ \vec{b}=2\vec{i}+3\vec{j}-\vec{k}$ and $\left(\vec{x},\ 2\vec{i}-3\vec{j}+4\vec{k}\right)=51$.

3.6. Find the coordinates of the vector $\vec{x}$, if it is orthogonal to the vectors $\vec{a}(2,\ 3,\ -1),\ \vec{b}(1,\ -1,\ 3)$ and $|\vec{x}|=1$.

3.7. Find the unit vector $\vec{d}$, coplanar to the vectors $\vec{a}(2,\ -1,\ 3)$ and $\vec{b}(4,\ 2,\ 0)$ and orthogonal to the vector $\vec{c}(1,1,1)$.

3.8. Calculate the area of a parallelogram whose sides are the vectors $\vec{a}=\vec{m}+2\vec{n}$ and $\vec{b}=\vec{m}-3\vec{n}$, if $|\vec{m}|=5,\ |\vec{n}|=3,\ (\vec{m}\wedge\vec{n})=\dfrac{\pi}{6}$.

3.9. Calculate the sine of the angle between the diagonals of a parallelogram whose sides are vectors $\vec{a}=2\vec{i}+\vec{j}+\vec{k},\ \ \vec{b}=\vec{i}-3\vec{j}+\vec{k}$.

3.10. Calculate the height of the parallelepiped built on the vectors $\vec{a}=3\vec{i}+2\vec{j}-5\vec{k},\ \vec{b}=\vec{i}-\vec{j}+4\vec{k},\ \vec{c}=\vec{i}-3\vec{j}+\vec{k}$, if the parallelogram built on the vectors $\vec{a}$ and $\vec{b}$ is taken as the base.

Task 4

4.1. Write the equation of a straight line passing through the origin perpendicular to the straight line $2x-6y+13=0$.

4.2. Find the angle between the line $2x+3y-1=0$ and the line passing through the points $M_1(-1;\ 2),\ M_2(0;\ 3)$.

4.3. Find the equation of a line passing through the point $M(-1;\ 4)$ parallel to the line $2x+3y-4=0$.

4.4. Given a triangle with vertices at the points $A(-1,\ 2)$, $B(0,1)$ and $C(1,\ 4)$. Write the equation of a straight line passing through the vertex A parallel to the opposite side.

4.5. At what value of the parameter α are the lines $(3\alpha+2)x+(1-4\alpha)y+8=0$ and $(5\alpha-2)x+(\alpha+4)y-7=0$ mutually perpendicular?

4.6. Triangle vertices $A(3,\ 5)$, $B(-3,\ 3)$ and $C(5,\ -8)$ are given. Determine the length of the median drawn from the vertex C.

4.7. For what values of α are the lines $ax-2y-1=0$ and $6x-4y-3=0$:

a) parallel; b) have one common point?

4.8. Write the equation of a straight line passing through a point $M(4;\ 3)$ perpendicular to the vector $\overrightarrow{M_1M_2}$, if $M_1(0,-2)$, $M_2(3,5)$.

4.9. Given a triangle with vertices at the points $M_1(2, 5)$, $M_2(-1, 3)$ and $M_3(0, 0)$. Write the equation for the median drawn from the vertex M_3.

4.10. Find the equation of the line passing through the point $M_1(-1, 2)$ perpendicular to the line connecting the points $M_2(2, 3)$ and $M_3(0, -1)$.

Task 5

Given the coordinates of the pyramid vertices $A_1A_2A_3A_4$. It is required to find: **1)** the length of the edge A_1A_2; **2)** the angle between the edges A_1A_2 and A_1A_4; **3)** the area of the face $A_1A_2A_3$; **4)** the volume of the pyramid; **5)** the equation of the straight line A_1A_4; **6)** the equation of the plane $A_1A_2A_3$; **7)** the angle between the edge A_1A_4 and the face $A_1A_2A_3$; **8)** the equation of the height dropped from the vertex A_4 to the face $A_1A_2A_3$. Make a drawing.

5.1.	$A_1(3,3,9)$,	$A_2(6,9,1)$,	$A_3(1,7,3)$,	$A_4(8,5,8)$.
5.2.	$A_1(3,5,4)$,	$A_2(5,8,3)$,	$A_3(1,9,9)$,	$A_4(6,4,8)$.
5.3.	$A_1(2,4,3)$,	$A_2(7,6,3)$,	$A_3(4,9,3)$,	$A_4(3,6,7)$.
5.4.	$A_1(9,5,5)$,	$A_2(-3,7,1)$,	$A_3(5,7,8)$,	$A_4(6,9,2)$.
5.5.	$A_1(0,7,1)$,	$A_2(4,1,5)$,	$A_3(4,6,3)$,	$A_4(3,9,8)$.
5.6.	$A_1(5,5,4)$,	$A_2(3,8,4)$,	$A_3(3,5,10)$,	$A_4(5,8,2)$.
5.7.	$A_1(6,1,1)$,	$A_2(4,6,6)$,	$A_3(4,2,0)$,	$A_4(1,2,6)$.
5.8.	$A_1(7,5,3)$,	$A_2(9,4,4)$,	$A_3(4,5,7)$,	$A_4(7,9,6)$.
5.9.	$A_1(6,6,2)$,	$A_2(5,4,7)$,	$A_3(2,4,7)$,	$A_4(7,3,0)$.
5.10.	$A_1(1,-3,1)$,	$A_2(-3,2,-3)$,	$A_3(-3,-3,3)$,	$A_4(-2,0,-4)$.

Task 6

Construct a curve on the plane, bringing its equation to the canonical form.

6.1. $x^2+8x+2y+20=0$. **6.2.** $3x^2-4y^2+18x+15=0$. **6.3.** $x^2+2y^2-2x+8y+7=0$.

6.4. $x^2+8x+y+15=0$. **6.5.** $x^2+y^2+4x-10y+20=0$. **6.6.** $5x^2+9y-30x+18y+9=0$.

6.7. $4x^2+9y^2-40x+36y+100=0$. **6.8.** $9x^2-16y^2-5x-64y-127=0$. **6.9.** $2x^2+8x-y+12=0$.

6.10. $x^2+4y^2-6y+3=0$.

Task 7

Construct a surface, bringing its equation to the canonical form.

7.1. a) $z=1-x^2-y^2$; b) $z=4-x^2$.

7.2. a) $x^2+2x+2y^2+4z^2=0$; b) $y^2+5y+z=4$.

7.3. a) $x^2+y^2+4z^2+6x=0$; b) $x^2+z^2=2z$.

7.4. a) $2y^2+z^2=1-x$; b) $xy=4$.

7.5. a) $9x^2+4y^2-8y-z^2=32$; b) $x^2-y^2-6x=0$.

7.6. a) $x^2-2y^2+z^2+2z=0$; b) $z^2+4z-6y-20=0$.

7.7. a) $x^2+y^2+z^2-3x+5y-4z=0$; b) $y^2=4x+1$.

7.8. a) $z=2+x^2+y^2$; b) $z=1-x^2$.

7.9. a) $36x^2+16y^2-9z^2+18z=9$; b) $z^2-2z-8x-7=0$.

7.10. a) $x^2-y^2-z^2=0$; b) $y^2=4x-2$.

TYPICAL CALCULATION No. 2

FUNCTION LIMIT. DERIVATIVE AND ITS APPLICATIONS TO RESEARCHING FUNCTIONS AND PLANTING

Task 1

Find the limits for the functions given below without using L'Hôpital's rule.

1.1. a) $\lim\limits_{x\to -3}\dfrac{x^2+x-6}{x^2+7x+12}$. b) $\lim\limits_{x\to\infty}\dfrac{5x^2+2x+1}{3x^3+3x^2-2}$. c) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x\sin x}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x-3}\right)^{x+2}$.

1.2. a) $\lim\limits_{x\to 2}\dfrac{2x^2-x-6}{x^2-3x+2}$. b) $\lim\limits_{x\to\infty}\dfrac{8x^4-2x^3+1}{5x^3+4x+3}$. c) $\lim\limits_{x\to\infty}\dfrac{1-\cos x}{x^2}$. d) $\lim\limits_{x\to 0}(1-4x)^{\frac{1-x}{x}}$.

1.3. a) $\lim\limits_{x\to -1}\dfrac{5x^2+x-4}{3x^2+5x+2}$. b) $\lim\limits_{x\to\infty}\dfrac{6x^5+4x-12}{3x^6-4x^2+1}$. c) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos^2 x}{x^2}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{3x+4}{3x+2}\right)^{x+2}$.

1.4. a) $\lim\limits_{x\to 1}\dfrac{2x^2+5x-7}{3x^2-x-2}$. b) $\lim\limits_{x\to\infty}\dfrac{5x^3+x^2-6}{2x^4-x-12}$. c) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x\,\mathrm{tg}\,x}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-2}\right)^{x}$.

1.5. a) $\lim\limits_{x\to -1}\dfrac{x^2+3x+2}{3x^2+4x+1}$. b) $\lim\limits_{x\to\infty}\dfrac{x^4-8x+1}{7x^5+4x^2+5}$. c) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos^3 x}{x^2}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^{x}$.

1.6. a) $\lim\limits_{x\to 2}\dfrac{2x^2-x-10}{x^2-x-2}$. b) $\lim\limits_{x\to\infty}\dfrac{2x^3-6x-5}{5x^2-x-1}$. c) $\lim\limits_{x\to 0}\dfrac{\mathrm{tg}^2\dfrac{x}{2}}{x}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2-1}\right)^{x^2}$.

1.7. a) $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{\dfrac{x}{2}-1}$. b) $\lim\limits_{x\to\infty}\dfrac{(x+1)^3-(x-1)^3}{(x+1)^2+(x+1)^2}$. c) $\lim\limits_{x\to 0}\dfrac{1-\cos 8x}{1-\cos 4x}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{3x-2}{3x+2}\right)^{2x}$.

1.8. a) $\lim\limits_{x\to 4}\dfrac{2x^2-9x+4}{x^2+x-20}$. b) $\lim\limits_{x\to\infty}\dfrac{2x^4-3x^2+1}{4x^6+6x^3-3}$. c) $\lim\limits_{x\to 0}\dfrac{x^2\mathrm{ctg}\,2x}{\sin 2x}$. d) $\lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}}$.

1.9. a) $\lim\limits_{x\to -2}\dfrac{x^2+7x+10}{2x^2+9x+10}$. b) $\lim\limits_{x\to\infty}\dfrac{2x^4+5x^2-3}{4x^6+6x^3-3}$. c) $\lim\limits_{x\to 0}\dfrac{1-\cos 6x}{1-\cos 2x}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+1}\right)^{x}$.

1.10. a) $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{2x^2-x-1}$. b) $\lim\limits_{x\to\infty}\dfrac{4+5x^2-4x^5}{8-6x-x^5}$. c) $\lim\limits_{x\to 0}\dfrac{x^2}{\sin^2\frac{x}{3}}$. d) $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-2}\right)^{2x-1}$.

Task 2

Investigate these functions for continuity and indicate the type of discontinuity points;

in condition "b" additionally pl;ot a graph of the function.

2.1. a) $f(x)=\frac{\ln(1+x)}{x^2}$. b) $f(x)=\begin{cases} x^2-1 \text{ if } -\infty<x\le 1; \\ \frac{2}{x} \quad \text{ if } \ 1<x<4; \\ x-3 \ \text{ if } \ x\ge 4. \end{cases}$

2.2. a) $f(x)=\operatorname{arctg}\frac{1}{x}$. b) $f(x)=\begin{cases} x^2 \ \text{ if } \ -\infty<x\le 0; \\ \sin x \text{ if } \ 1<x<\frac{\pi}{6}; \\ \frac{1}{2} \quad \text{ if } \ x\ge\frac{\pi}{6}. \end{cases}$

2.3. a) $f(x)=3^{\frac{1}{x-2}}$. b) $f(x)=\begin{cases} \ln x \ \text{ if } \ 0<x\le 1; \\ x-1 \ \text{ if } \ 1<x\le 3; \\ x^2-3 \text{ if } \ x>3. \end{cases}$

2.4. a) $f(x)=\frac{1}{1-e^{1-x}}$. b) $f(x)=\begin{cases} \operatorname{tg} x \quad \text{ if } \ 0<x\le\frac{\pi}{4}; \\ \frac{2\pi}{x} \quad \text{ if } \ \frac{\pi}{4}<x<\pi; \\ \sin x+2 \text{ if } \ x\ge\pi. \end{cases}$

2.5. a) $f(x)=\frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1}$. b) $f(x)=\begin{cases} x+1 \ \text{ if } \ -\infty<x\le 1; \\ 3^x \quad \text{ if } \ 0<x\le 2; \\ 6-x \ \text{ if } \ x>2. \end{cases}$

2.6. a) $f(x)=\frac{|x-2|}{x-2}$. b) $f(x)=\begin{cases} 2\sqrt{x} & \text{if} \quad 0<x\le 1; \\ x^2+2 & \text{if} \quad 1<x\le 2; \\ \frac{2}{x}+4 & \text{if} \quad x>2. \end{cases}$

2.7. a) $f(x)=\frac{x^2-3x+2}{x-x^3}$. b) $f(x)=\begin{cases} x^2+1 & \text{if} \quad -\infty<x\le 1; \\ \frac{2}{x} & \text{if} \quad 1<x\le 4; \\ x-2 & \text{if} \quad x>4. \end{cases}$

2.8. a) $f(x)=\frac{1}{1-x}-\frac{2}{1-x^2}$. b) $f(x)=\begin{cases} x+1 & \text{if} \quad -\infty<x\le 3; \\ 3x-7 & \text{if} \quad 3<x\le 4; \\ 3+\sqrt{x} & \text{if} \quad x>4. \end{cases}$

2.9. a) $f(x)=\frac{x^2-5x+6}{x^2-2x}$. b) $f(x)=\begin{cases} \cos x & \text{if} \quad x\le 0; \\ 1-x & \text{if} \quad 0<x\le 3; \\ x^2-5 & \text{if} \quad x>3. \end{cases}$

2.10. a) $f(x)=\frac{\sin(x-3)}{x^2-4x+3}$. b) $f(x)=\begin{cases} 0 & \text{if} \quad x<0; \\ \operatorname{tg} x & \text{if} \quad 0\le x\le \frac{\pi}{4}; \\ \frac{4}{\pi}x & \text{if} \quad x>\frac{\pi}{4}. \end{cases}$

Task 3

Find the derivatives of the functions given below.

3.1. a) $y=\sqrt{x}\arcsin\sqrt{x}+\sqrt{1-x}$; b) $y=x^{\arcsin x}$; c) $x^4-6x^2y^2+9y^4-5x^2+15y^2-100=0$.

3.2. a) $y=\ln\operatorname{tg}\frac{2x+1}{4}$; b) $y=x^{\frac{1}{\ln x}}$; c) $x^y-y^x=0$.

3.3. a) $y=\ln\sqrt{\frac{1+\sin x}{1-\sin x}}$; b) $y=x^x$; c) $e^x+e^y-2^{xy}-3=0$.

3.4. a) $y=\ln\left(3x^2+\sqrt{9x^4+1}\right)$; b) $y=x^{\ln x}$; c) $\sin\left(y-x^2\right)-\ln\left(y-x^2\right)+2\sqrt{y-x^2}-3=0$.

3.5. a) $y=\arcsin\frac{2x^3}{1+x^6}$; b) $y=x^{\sin x}$; c) $\frac{y}{x}+e^{\frac{y}{x}}-\sqrt[3]{\frac{y}{x}}=0$.

3.6. a) $y=\operatorname{arctg}\sqrt{\frac{1-x}{1+x}}$; b) $y=(\sin x)^{\cos x}$; c) $x^2\sin y+y^3\cos x-2x-3y+1=0$.

3.7. a) $y=\arcsin\frac{\sin x}{\sqrt{1+\sin^2 x}}$; b) $y=(x+1)^{\frac{2}{x}}$; c) $\frac{x^2}{25}+\frac{y^2}{9}=1$.

3.8. a) $y=\ln\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$; b) $y=x^2e^{x^2}\sin 2x$; c) $x^4+y^4=x^2y^2$.

3.9. a) $y=e^x-\sin e^x\cos^3 e^x-\sin^3 e^x\cos e^x$; b) $y=x^2e^{x^2}\ln x$; c) $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

3.10. a) $y=\operatorname{arctg}(x+1)+\frac{x+1}{x^2+2x+2}$; b) $y=(x+1)^{\frac{2}{x}}$; c) $2y\ln y=x$.

Task 4

Find the second order derivatives for the functions given below:

4.1. $y = \cos^2 x$.

4.2. $y = \mathrm{arctg} x^3$.

4.3. $y = \log_2 \sqrt[3]{1-x^4}$.

4.4. $y = e^{-x^2}$.

4.5. $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$.

4.6. $y = -\dfrac{22x}{x+5}$.

4.7. $y = \dfrac{1}{4}x^2(2\ln x - 3)$.

4.8. $y = \dfrac{1}{3}x^2 \cdot \sqrt{1-x^2} + \dfrac{2}{3} \cdot \sqrt{1-x^2} + x \arcsin x$.

4.9. $y = -\dfrac{1}{9}x \cdot \sin 3x - \dfrac{2}{27}\cos 3x$.

4.10. $y = \sin^2 x$.

Task 5

Find the first and the second order derivatives for the next functions given parametrically:

5.1. $x = t^2 + 2;\ y = \dfrac{1}{3}t^3 - 1$.

5.2. $x = \arcsin t;\ y = \sqrt{1-t^2}$.

5.3. $x = at^2;\ y = bt^3$.

5.4. $x = \cos t;\ y = \sin t$.

5.5. $x = a(t - \sin t);\ y = a(1 - \cos t)$.

5.6. $x = a\cos^2 t;\ y = a\sin^2 t$.

5.7. $x = \ln t;\ y = t^2 - 1$.

5.8. $x = \arcsin t;\ y = \ln(1 - t^2)$.

5.9. $x = at \cdot \cos t;\ y = at \cdot \sin t$.

5.10. $x = \arccos\sqrt{t};\ y = \sqrt{t - t^2}$.

Task 6

Using L'Hôpital's rule, find the limits for the functions:

6.1. a) $\lim\limits_{x\to -1}\dfrac{\sqrt[3]{2x+1}+1}{\sqrt{x+2}+x}$; b) $\lim\limits_{x\to 0}\dfrac{\ln x}{\operatorname{ctg} x}$.

6.2. a) $\lim\limits_{x\to 0}\dfrac{1-\cos\alpha x}{1-\cos\beta x}$; b) $\lim\limits_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right)$.

6.3. a) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$; b) $\lim\limits_{x\to\infty}(\pi-2\operatorname{arctg} x)\ln x$.

6.4. a) $\lim\limits_{x\to\infty}\dfrac{x-\sin x}{x^3}$; b) $\lim\limits_{x\to -1+0}\dfrac{\operatorname{tg}\frac{\pi x}{2}}{\ln(1+x)}$.

6.5. a) $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^{-2ax}}{\ln(1+x)}$; b) $\lim\limits_{x\to 1+0}\dfrac{\ln(x-1)}{\operatorname{ctg}\pi x}$.

6.6. a) $\lim\limits_{x\to 1}\dfrac{x^3-2x^2-x+2}{x^3-7x+6}$; b) $\lim\limits_{x\to 0}\arcsin x\cdot\operatorname{ctg} x$.

6.7. a) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$; b) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^\alpha}\ (\alpha>0)$.

6.8. a) $\lim\limits_{x\to 0}\dfrac{e^{\alpha x}-e^{-\alpha x}}{\sin x}$; b) $\lim\limits_{x\to\infty}\dfrac{x^{100}}{e^x}$.

6.9. a) $\lim\limits_{x\to\infty}\dfrac{\frac{\pi}{4}-\operatorname{arctg}\left(1-\frac{1}{x}\right)}{\sin\frac{1}{x}}$; b) $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\dfrac{\pi x}{2}$.

6.10. a) $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{\operatorname{tg} x}$; b) $\lim\limits_{x\to -1}(1+x)\operatorname{tg}\dfrac{\pi x}{2}$.

Task 7

Write the Taylor formula of the third order with a remainder term in the Lagrange form for a given function at a point x_0.

7.1. $xe^{2x}, \; x_0 = -1$.

7.2. $\frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \; x_0 = 0$.

7.3. $e^x, \; x_0 = -1$.

7.4. $4^x, \; x_0 = 0$.

7.5. $\sqrt{x}, \; x_0 = 4$.

7.6. $x^{10} - 3x^6 + x^2 + 2, \; x_0 = 1$.

7.7. $\frac{1}{x+8}, \; x_0 = 0$.

7.8. $x\cos x, \; x_0 = 0$.

7.9. $\frac{x}{x-1}, \; x_0 = 2$.

7.10. $e^{\sin x}, \; x_0 = 0$.

Task 8

Explore a function and plot its graph.

8.1. $y = \frac{1-x^2}{x^2}$.

8.2. $y = \frac{x}{(1+x)^3}$.

8.3. $y = \frac{4x^2+1}{x}$.

8.4. $y = \frac{x^3}{x^2-1}$.

8.5. $y = \frac{x^3}{2(1+x)^2}$.

8.6. $y = \frac{x^3+2}{2x}$.

8.7. $y = \frac{4x}{4+x^2}$.

8.8. $y = \frac{x^2-1}{x^2+1}$.

8.9. $y = \frac{x^2}{x-1}$.

8.10. $y = \frac{4x^3+5}{x}$.

CONTROL WORK No. 1

Task 1. **a) Check the compatability of the system of linear equations and solve it using the Cramer formulas and the matrix method.**

b) Investigate the system for compatibility and, in case of compatibility, solve it using the Gauss method.

1.1. a)
$$\begin{cases} x_1 + x_2 - x_3 = 6, \\ 2x_1 + 3x_2 - 4x_3 = 21, \\ 7x_1 - x_2 - 3x_3 = 6. \end{cases}$$
b)
$$\begin{cases} 6x_1 + x_2 - x_3 = 1, \\ x_1 - x_2 + 2x_3 = 3, \\ 8x_1 - x_2 + 3x_3 = 7. \end{cases}$$

1.2. a)
$$\begin{cases} 3x_1 - 3x_2 + 4x_3 = 7, \\ x_1 + x_2 - 5x_3 = -6, \\ 2x_1 - x_2 + x_3 = 2. \end{cases}$$
b)
$$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 1, \\ x_1 - 2x_2 + x_3 = -3, \\ 5x_1 - 3x_2 + 7x_3 = 2. \end{cases}$$

1.3. a)
$$\begin{cases} x_1 + 2x_2 - x_3 = 2, \\ 2x_1 - 3x_2 + 2x_3 = 2, \\ 3x_1 + x_2 + x_3 = 8. \end{cases}$$
b)
$$\begin{cases} x_1 - 2x_2 + x_3 = 6, \\ 2x_1 + x_2 - x_3 = 2, \\ 3x_1 + 4x_2 - 3x_3 = -2. \end{cases}$$

1.4. a)
$$\begin{cases} 2x_1 - 3x_2 + 5x_3 = 11, \\ 3x_1 - x_2 + 5x_3 = 10, \\ x_1 + 2x_2 - 4x_3 = -7. \end{cases}$$
b)
$$\begin{cases} 4x_1 - 2x_2 - x_3 = 1, \\ x_1 + x_2 + 3x_3 = 3, \\ x_1 - 5x_2 - 10x_3 = 0. \end{cases}$$

1.5. a)
$$\begin{cases} 4x_1 - x_2 - 3x_3 = 1, \\ 3x_1 + 6x_2 + 2x_3 = 4, \\ 2x_1 + 4x_2 + x_3 = 4. \end{cases}$$
b)
$$\begin{cases} x_1 + x_2 - x_3 = 2, \\ 3x_1 - 3x_2 + 2x_3 = 5, \\ 9x_1 - 3x_2 + x_3 = 16. \end{cases}$$

1.6. a)
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6, \\ 2x_1 + 3x_2 - x_3 = 4, \\ 3x_1 + x_2 - 4x_3 = 0. \end{cases}$$
b)
$$\begin{cases} x_1 + 2x_2 - x_3 = 0, \\ 5x_1 + x_2 - 7x_3 = 0, \\ x_1 + 2x_2 + 5x_3 = 1. \end{cases}$$

1.7. a)
$$\begin{cases} 2x_1 + x_2 = 3, \\ x_1 + x_2 = 1, \\ 3x_1 + x_2 + 2x_3 = 0. \end{cases}$$
b)
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 1, \\ -5x_1 + x_2 = 1, \\ -3x_1 + 4x_2 - 4x_3 = 2. \end{cases}$$

1.8. a) $\begin{cases} x_1 - x_2 - 3x_3 = 13, \\ 2x_1 + x_2 - x_3 = 0, \\ 3x_1 - 2x_2 + 4x_3 = -15. \end{cases}$ b) $\begin{cases} x_1 - x_2 - 3x_3 = 0, \\ -2x_1 + 2x_2 + 5x_3 = 1, \\ 3x_1 - 3x_2 - 9x_3 = 0. \end{cases}$

1.9. a) $\begin{cases} 2x_1 + 3x_2 + 5x_3 = 12, \\ x_1 - 4x_2 + 3x_3 = -22, \\ 3x_1 - x_2 - 2x_3 = 0. \end{cases}$ b) $\begin{cases} x_1 + 3x_2 = 1, \\ 2x_1 + 6x_2 + x_3 = 1, \\ 3x_1 + 9x_2 + x_3 = 2. \end{cases}$

1.10. a) $\begin{cases} 2x_1 + 3x_2 - x_3 = 4, \\ x_1 + 2x_2 + 2x_3 = 5, \\ 3x_1 + 4x_2 - 3x_3 = 2. \end{cases}$ b) $\begin{cases} x_1 + 4x_3 = 1, \\ x_2 - 3x_3 = 2, \\ x_1 + x_2 + 7x_3 = 0. \end{cases}$

1.11. a) $\begin{cases} x_2 + 3x_3 = -6, \\ x_1 - 2x_2 - x_3 = 5, \\ 3x_1 + 4x_2 - 2x_3 = 13. \end{cases}$ b) $\begin{cases} x_1 + 5x_2 - 4x_3 = -3, \\ x_2 + 3x_3 = 2, \\ x_1 + 7x_2 + 2x_3 = 1. \end{cases}$

1.12. a) $\begin{cases} 4x_1 + 2x_2 - x_3 = 12, \\ x_1 + 2x_2 + x_3 = 7, \\ x_2 - x_3 = -1. \end{cases}$ b) $\begin{cases} 3x_1 + 2x_2 - 4x_3 = 8, \\ 2x_1 + 4x_2 - 5x_3 = 11, \\ x_1 - 2x_2 + x_3 = 1. \end{cases}$

1.13. a) $\begin{cases} 2x_1 + x_2 - x_3 = 0, \\ 3x_2 - 4x_3 = -6, \\ x_1 + x_3 = 1. \end{cases}$ b) $\begin{cases} x_1 + x_2 + x_3 = 1, \\ x_1 - x_2 + 2x_3 = -5, \\ 2x_1 + 3x_3 = -2. \end{cases}$

1.14. a) $\begin{cases} x_1 + 2x_2 + x_3 = 8, \\ 3x_1 - 2x_2 - 3x_3 = -5, \\ 3x_1 - 4x_2 + 5x_3 = 10. \end{cases}$ b) $\begin{cases} 2x_1 - x_2 + 4x_3 = 15, \\ 3x_1 - x_2 + x_3 = 8, \\ 5x_1 - 2x_2 + 5x_3 = 0. \end{cases}$

1.15. a) $\begin{cases} 2x_1 - x_2 = -1, \\ x_1 - 2x_2 - x_3 = -2, \\ x_2 + x_3 = -2. \end{cases}$ b) $\begin{cases} 3x_1 - 3x_2 + 2x_3 = 2, \\ 4x_1 - 5x_2 + 2x_3 = 1, \\ x_1 - 2x_2 = 5. \end{cases}$

1.16. a) $\begin{cases} 2x_1 - 3x_2 - x_3 = -6, \\ 3x_1 + 4x_2 + 3x_3 = -5, \\ x_1 + x_2 + x_3 = -2. \end{cases}$ b) $\begin{cases} 3x_1 + 2x_2 - 4x_3 = 8, \\ 2x_1 + 4x_2 - 5x_3 = 1, \\ 5x_1 + 6x_2 - 9x_3 = 2. \end{cases}$

1.17. a) $\begin{cases} 3x_1 + 2x_2 + 5x_3 = -10, \\ 2x_1 + 5x_2 - 3x_3 = 6, \\ x_1 + 3x_2 - x_3 = -6. \end{cases}$ b) $\begin{cases} 3x_1 + x_2 + 2x_3 = -3, \\ 2x_1 + 2x_2 + 5x_3 = 5, \\ 5x_1 + 3x_2 + 7x_3 = 1. \end{cases}$

1.18. a) $\begin{cases} x_1 + 2x_2 + x_3 = 8, \\ -2x_1 + 3x_2 - 3x_3 = -5, \\ 3x_1 - 4x_2 + 5x_3 = 10. \end{cases}$ b) $\begin{cases} 4x_1 - 7x_2 - 2x_3 = 0, \\ 2x_1 - 3x_2 - 4x_3 = 6, \\ 2x_1 - 4x_2 + 2x_3 = 2. \end{cases}$

1.19. a) $\begin{cases} 3x_1 + 4x_2 + 2x_3 = 8, \\ 2x_1 - 4x_2 - 3x_3 = -1, \\ x_1 + x_2 + x_3 = 0. \end{cases}$ b) $\begin{cases} 5x_1 - 9x_2 - 4x_3 = 6, \\ x_1 - 7x_2 - 5x_3 = 1, \\ 4x_1 - 2x_2 + x_3 = 2. \end{cases}$

1.20. a) $\begin{cases} 2x_1 + 3x_2 - x_3 = 5, \\ 3x_1 - x_2 + x_3 = 4, \\ x_1 + x_2 + x_3 = 6. \end{cases}$ b) $\begin{cases} x_1 - 5x_2 + x_3 = 3, \\ 3x_1 + 2x_2 - x_3 = 7, \\ 4x_1 - 3x_2 = 1. \end{cases}$

1.21. a) $\begin{cases} 2x_1 - x_2 = 0, \\ x_1 + 2x_2 - x_3 = -2, \\ x_2 + x_3 = -5. \end{cases}$ b) $\begin{cases} 5x_1 - 5x_2 - 4x_3 = -3, \\ x_1 - x_2 + 5x_3 = 1, \\ 4x_1 - 4x_2 - 9x_3 = 0. \end{cases}$

1.22. a) $\begin{cases} x_2 + 3x_3 = -6, \\ x_1 - 2x_2 - x_3 = 5, \\ 3x_1 + 4x_2 - 2x_3 = 13. \end{cases}$ b) $\begin{cases} 7x_1 - 2x_2 - x_3 = 2, \\ 6x_1 - 4x_2 - 5x_3 = 3, \\ x_1 + 2x_2 + 4x_3 = 5. \end{cases}$

1.23. a) $\begin{cases} x_1 + 3x_2 - x_3 = -1, \\ x_1 - x_2 + 5x_3 = 9, \\ 2x_1 + x_2 - 2x_3 = 3. \end{cases}$ b) $\begin{cases} 4x_1 - 3x_2 + x_3 = 3, \\ x_1 + x_2 - x_3 = 4, \\ 3x_1 - 4x_2 + 2x_3 = 2. \end{cases}$

1.24. a) $\begin{cases}5x_1+8x_2-x_3=7,\\ 2x_1-3x_2+2x_3=9,\\ x_1+2x_2+3x_3=1.\end{cases}$ b) $\begin{cases}3x_1+x_2+2x_3=1,\\ 2x_1+2x_2-3x_3=9,\\ x_1-x_2+x_3=2.\end{cases}$

1.25. a) $\begin{cases}2x_1-x_2+5x_3=4,\\ 5x_1+2x_2+13x_3=2,\\ 3x_1-x_2+5x_3=0.\end{cases}$ b) $\begin{cases}2x_1+3x_2+4x_3=5,\\ x_1+x_2+5x_3=6,\\ 3x_1+4x_2+9x_3=0.\end{cases}$

1.26. a) $\begin{cases}4x_1+x_2-x_3=6,\\ x_1-x_2+2x_3=-3\\ 2x_1-7x_2+x_3=0.\end{cases}$ b) $\begin{cases}5x_1+6x_2-2x_3=2,\\ 2x_1+3x_2-x_3=9,\\ 3x_1+3x_2-x_3=1.\end{cases}$

1.27. a) $\begin{cases}2x_1-x_2=0,\\ x_1+2x_2-x_3=1,\\ x_2+x_3=0.\end{cases}$ b) $\begin{cases}4x_1-9x_2+5x_3=1,\\ 7x_1-4x_2+x_3=11,\\ 3x_1+5x_2-4x_3=5.\end{cases}$

1.28. a) $\begin{cases}3x_1+4x_2+2x_3=8,\\ x_1+5x_2+2x_3=5,\\ 2x_1+3x_2+4x_3=3.\end{cases}$ b) $\begin{cases}3x_1+4x_2+x_3=2,\\ x_1+5x_2-3x_3=4,\\ 2x_1-x_2+4x_3=5.\end{cases}$

1.29. a) $\begin{cases}x_1+x_2-x_3=-2,\\ 2x_1+4x_2+3x_3=3,\\ 3x_1-2x_2+5x_3=13.\end{cases}$ b) $\begin{cases}2x_1+8x_2-7x_3=0,\\ 2x_1-5x_2+6x_3=1,\\ 4x_1+3x_2-x_3=7.\end{cases}$

1.30. a) $\begin{cases}4x_1+2x_2-x_3=0,\\ x_1+2x_2+x_3=8,\\ x_2-x_3=-3.\end{cases}$ b) $\begin{cases}3x_1-5x_2+3x_3=4,\\ x_1+2x_2+x_3=8,\\ 2x_1-7x_2+2x_3=1.\end{cases}$

Task 2. The coordinates for the points A, B, C are given. It is required to find: 1) $\text{proj}_{\overline{AC}}\overrightarrow{AB}$; **2) the area of a triangle with vertices at points A, B, C.**

2.1. A(7,1,4), B(9,–2,0), C(0,3,–3).

2.2. A(3,1,4), B(–3,–1,0), C(2,1,–3).

2.3. A(2,1,0),	B(3,–1,–4),	C(0,2,–2).
2.4. A(3,–1,–1),	B(3,1,4),	C(1,0,5).
2.5. A(2,1,–1),	B(7,–1,3),	C(0,3,3).
2.6. A(2,–3,7),	B(–3,–1,5),	C(9,0,1).
2.7. A(7,–3,4),	B(3,2,–1),	C(4,1,1).
2.8. A(1,1,0),	B(2,1,–4),	C(0,1,0).
2.9. A(1,–1,4),	B(2,3,–4),	C(1,0,–5).
2.10. A(2,–4,7),	B(8,1,0),	C(–1,–3,0).
2.11. A(1,–1,0),	B(0,1,7),	C(–1,–2,–3).
2.12. A(0,9,–3),	B(1,3,4),	C(0,2,–5).
2.13. A(1,–1,3),	B(2,–2,4),	C(1,0,1).
2.14. A(2,–2,–3),	B(–1,–4,7),	C(0,4,–3).
2.15. A(1,0,0),	B(–3,1,–1),	C(1,–2,–3).
2.16. A(1,3,7),	B(7,3,–5),	C(–1,–4,0).
2.17. A(1,–1,1),	B(0,1,0),	C(1,4,–5).
2.18. A(2,–2,3),	B(1,–1,4),	C(0,1,–1).
2.19. A(2,0,–1),	B(1,–1,1),	C(0,1,7).
2.20. A(1,–1,3),	B(2,1,–4),	C(0,1,0).
2.21. A(1,–2,2),	B(2,0,1),	C(1,4,–7).
2.22. A(1,2,–3),	B(2,–1,4),	C(2,3,–4).
2.23. A(7,9,–3),	B(1,0,–1),	C(0,3,0).
2.24. A(1,–2,0),	B(2,4,–1),	C(7,1,0).
2.25. A(3,–1,4),	B(–4,2,3),	C(0,1,–1).

2.26. A(6,–3,0),	B(3,0,1),	C(2,–4,3).
2.27. A(4,5,–1),	B(6,–4,2),	C(0,3,–1).
2.28. A(3,–1,2),	B(3,6,–4),	C(0,1,–1).
2.29. A(1,1,–1),	B(3,4,0),	C(0,5,–2).
2.30. A(1,0,2),	B(3,4,7),	C(5,–1,1).

Task 3. **Find the angle (in degrees) between the line** $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z+1}{3}$ **and the plane, passing through points** M_1, M_2, M_3**.**

3.1.	M_1(1,–3,4),	M_2(0,–2,–1),	M_3(1,1,–1).
3.2.	M_1(1,1,4),	M_2(–2,1,1),	M_3(1,3,6).
3.3.	M_1(1,2,–1),	M_2(–1,0,4),	M_3(–2,–1,1).
3.4.	M_1(1,2,3),	M_2(4,–1,–2),	M_3(4,0,3).
3.5.	M_1(1,3,–1),	M_2(–3,1,–9),	M_3(1,0,–7).
3.6.	M_1(1,–2,–1/2),	M_2(2,1,3),	M_3(0,–1,–1).
3.7.	M_1(1,1,4),	M_2(2,–1,0),	M_3(3,2,1).
3.8.	M_1(–13,3,2),	M_2(–3,–2,–4),	M_3(0,0,–3).
3.9.	M_1(1,–1,–3),	M_2(0,6,1),	M_3(2,2,–2).
3.10.	M_1(2,3,–10),	M_2(1,–1,–9),	M_3(0,–1,–4).
3.11.	M_1(1,1,4),	M_2(2,0,2),	M_3(0,3,3).
3.12.	M_1(2,1,–3),	M_2(1,1,0),	M_3(–1,2,7).
3.13.	M_1(1,0,1),	M_2(0,0,2),	M_3(1,1,1).
3.14.	M_1(–5,–1,1),	M_2(–2,0,1),	M_3(–1,1,0).
3.15.	M_1(2,1,3),	M_2(0,0,4),	M_3(1,1,1).

3.16.	$M_1(2,3,1)$,	$M_2(4,-4,-2)$,	$M_3(1,0,0)$.
3.17.	$M_1(-1,0,1)$,	$M_2(3,-2,-1)$,	$M_3(-4,-1,2)$.
3.18.	$M_1(2,-2,9)$,	$M_2(-2,0,1)$,	$M_3(-4,1,3)$.
3.19.	$M_1(1,2,-1)$,	$M_2(2,3,-10)$,	$M_3(0,4,1)$.
3.20.	$M_1(1,-2,1)$,	$M_2(0,-1,2)$,	$M_3(2,-1,-1)$.
3.21.	$M_1(1,-2,-5)$,	$M_2(2,3,2)$,	$M_3(-1,0,5)$.
3.22.	$M_1(1,3,4)$,	$M_2(0,1,2)$,	$M_3(2,5,0)$.
3.23.	$M_1(1,-1,0)$,	$M_2(-3,-4,1)$,	$M_3(-1,-1,2)$.
3.24.	$M_1(-1,2,0)$,	$M_2(6,3,1)$,	$M_3(-15,0,2)$.
3.25.	$M_1(1,2,3)$,	$M_2(2,4,1)$,	$M_3(2,0,-3)$.
3.26.	$M_1(-1,1,0)$,	$M_2(3,-4,5)$,	$M_3(-2,0,2)$.
3.27.	$M_1(2,-3,5)$,	$M_2(1,-2,12)$,	$M_3(4,-1,7)$.
3.28.	$M_1(3,-1,2)$,	$M_2(4,-1,-1)$,	$M_3(2,0,2)$.
3.29.	$M_1(1,3,1)$,	$M_2(4,0,7)$,	$M_3(-2,1,2)$.
3.30.	$M_1(1,-1,1)$,	$M_2(5,4,-2)$,	$M_3(-1,-2,2)$.

Task 4. Simplify the equation of the curve and depict it in the figure

4.1. $x^2+2y^2-2x+8y+7=0$

4.2. $9x^2+y^2-36x+2y+28=0$

4.3. $x^2+8x+2y+20=0$

4.4. $4x^2+9y^2-40x-36y+100=0$

4.5. $x^2-y^2+2x+6y-12=0$

4.6. $x^2+y^2+4x-10y+20=0$

4.7. $9x^2+4y^2-54x-32y+109=0$

4.8. $25x^2-9y^2-150x-72y-144=0$

4.9. $2x^2+8x-y+12=0$

4.10. $9x^2+4y^2-18x=0$

4.11. $x^2-4y^2-4x+40=0$

4.12. $x^2-4y^2+6x+16y-11=0$

4.13. $9x^2+10y^2+40y-50=0$

4.14. $9x^2-16y^2-18x+64y+89=0$

4.15. $x^2+4y^2-6x+8y=3$

4.16. $x^2+4y^2+2x=0$

4.17. $4x^2-y^2-24x-6y+43=0$

4.18. $x=2y^2-12y+14$

4.19. $y^2+4y=2x$

4.20. $2x^2-5y^2+4x+40y-58=0$

4.21. $4x^2-9y^2+4x=0$

4.22. $x^2-8x+y+15=0$

4.23. $3x^2+4y^2+30x-24y+99=0$

4.24. $3x^2-4y^2+18x+15=0$

4.25. $9x^2-16y^2-54x-64y-127=0$

4.26. $y^2-6x+6y+27=0$

4.27. $5x^2+9y^2-30x+18y+9=0$

4.28. $4x^2+8x-y+7=0$

4.29. $y^2+2x-4y+14=0$

4.30. $x^2-5x-y+7=0$

Task 5. Find the limits for the functions given below:

5.1. a) $\lim\limits_{x\to\infty}\frac{3x^3-5x^2+2}{2x^3-4x^2-x}$; b) $\lim\limits_{x\to 0}\frac{1-\cos x}{x\sin 3x}$; c) $\lim\limits_{x\to\infty}\left(\frac{2x-1}{2x+1}\right)^x$; d) $\lim\limits_{x\to 2}\frac{\ln\left(x^2-3\right)}{x^2+3x-10}$.

5.2. a) $\lim\limits_{x\to -1}\frac{x^2+3x+2}{3x^2+4x+1}$; b) $\lim\limits_{x\to 0}\frac{\cos x-\cos 3x}{1-\sqrt{1-x^2}}$; c) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$; d) $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\operatorname{tg}x}$.

5.3. a) $\lim\limits_{x\to 0}\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+16}-4}$; b) $\lim\limits_{x\to\infty}\frac{5x^3+x^2-6}{2x^4-x+12}$; c) $\lim\limits_{x\to 0}\frac{x\sin 2x}{1-\cos 4x}$; d) $\lim\limits_{x\to\infty}\left(\cos\frac{1}{x^2}\right)^{x^2}$.

5.4. a) $\lim\limits_{x\to 3}\frac{x^3-27}{\sqrt{3x}-x}$; b) $\lim\limits_{x\to\infty}\left(\frac{2x+5}{2x+1}\right)^{5x}$; c) $\lim\limits_{x\to 0}\frac{\cos x-\cos^3 x}{3x^2}$; d) $\lim\limits_{x\to\infty}\frac{x+2\ln x}{x}$.

5.5. a) $\lim\limits_{x\to\infty}\frac{(x+1)^3-(x-1)^3}{(x+2)^2+(x+1)^2}$; b) $\lim\limits_{x\to 0}\frac{1-\cos 3x}{2x\operatorname{tg}4x}$; c) $\lim\limits_{x\to 0}\frac{\sqrt{1-x^2}-1}{e^{-x^2}-1}$; d) $\lim\limits_{x\to\frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x}$.

5.6. a) $\lim\limits_{x\to 4}\frac{x^3-64}{7x^2-27x-4}$; b) $\lim\limits_{x\to\infty}\left(\sqrt{x+2}-\sqrt{x}\right)$; c) $\lim\limits_{x\to 0}\frac{1-\cos 2x}{x\sin 3x}$; d) $\lim\limits_{x\to 1-0}(1-x)\ln(1-x)$.

5.7. a) $\lim\limits_{x\to\infty}\frac{3x^3-2x+1}{5x^2-x+2}$; b) $\lim\limits_{x\to 0}\frac{\cos x-\cos 2x}{2-\sqrt{4-x^2}}$; c) $\lim\limits_{x\to 5}\frac{2x^2-9x-5}{x^2-4x-5}$; d) $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\operatorname{tg}x}$.

5.8. a) $\lim\limits_{x\to 2}\dfrac{3x^2-x-10}{x^3-x-6}$; b) $\lim\limits_{x\to\infty}\left(\dfrac{3x+1}{3x+4}\right)^{x+1}$; c) $\lim\limits_{x\to 0}\dfrac{1-\cos 7x}{2x\,\mathrm{tg}3x}$; d) $\lim\limits_{x\to 0}x^2e^{\frac{1}{x^2}}$.

5.9. a) $\lim\limits_{x\to 1}\dfrac{2x^2+x-3}{3x^2-2x-1}$; b) $\lim\limits_{x\to\infty}\dfrac{11x^5-5x^2-1}{20x^4-4x+8}$; c) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\mathrm{tg}3x}$; d) $\lim\limits_{x\to 0}\left(3^x+x\right)^{\frac{2}{x}}$.

5.10. a) $\lim\limits_{x\to\infty}\dfrac{x^3+5x}{2x^4-3x^2+1}$; b) $\lim\limits_{x\to 0}\dfrac{\cos x-1}{\cos 2x-\cos 5x}$; c) $\lim\limits_{x\to\infty}\dfrac{1-e^{-\frac{1}{x}}}{\ln\left(1-\dfrac{1}{x}\right)}$; d) $\lim\limits_{x\to 2}\left(5-2x\right)^{\frac{x^2}{x-2}}$.

5.11. a) $\lim\limits_{x\to\infty}\dfrac{5x^2-3x+1}{7x^2+x-5}$; b) $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{\sqrt{5-x}-\sqrt{x+1}}$; c) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2-1}\right)^{x^2}$; d) $\lim\limits_{x\to\infty}x^2\sin\dfrac{3}{x}$.

5.12. a) $\lim\limits_{x\to\infty}\left(\sqrt{x^2-2x-1}-\sqrt{x^2-7x+3}\right)$; b) $\lim\limits_{x\to 3}\dfrac{4x^2-5x-21}{2x^2-3x-9}$; c) $\lim\limits_{x\to 0}\dfrac{\mathrm{tg}x-\sin x}{x^3}$; d) $\lim\limits_{x\to 1}\left(1-x\right)^{\ln x}$.

5.13. a) $\lim\limits_{x\to -4}\dfrac{5x^2+9x-44}{2x^2+5x-12}$; b) $\lim\limits_{x\to\infty}\dfrac{3x^3+x^2-1}{4x^4+x+3}$; c) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{3^x-1}$; d) $\lim\limits_{x\to\frac{\pi}{2}}\left(\sin x\right)^{\mathrm{tg}x}$.

5.14. a) $\lim\limits_{x\to\infty}\dfrac{x-x^2+3x^3}{4x^3-2x^2+1}$; b) $\lim\limits_{x\to 5}\left(\dfrac{1}{x-5}-\dfrac{5}{x^2-x-20}\right)$; c) $\lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^2-\sin x^2}$; d) $\lim\limits_{x\to 0}\left(x+e^{2x}\right)^{\frac{1}{x}}$.

5.15. a) $\lim\limits_{x\to 2}\dfrac{x^3-x-6}{3x^2-x-10}$; b) $\lim\limits_{x\to\infty}\dfrac{4+5x^2-4x^3}{8-6x+x^5}$; c) $\lim\limits_{x\to 0}\left(1-4x\right)^{\frac{1-x}{x}}$; d) $\lim\limits_{x\to 0}\dfrac{\ln\left(1+x^2\right)}{\cos 3x-e^{-x}}$.

5.16. a) $\lim\limits_{x\to\infty}\dfrac{4x^2-12x+17}{5x^2+3x+1}$; b) $\lim\limits_{x\to 4}\dfrac{2x^2-9x+4}{\sqrt{5-x}-\sqrt{x-3}}$; c) $\lim\limits_{x\to 0}\dfrac{\cos 5x-\cos 2x}{3x^2}$; d) $\lim\limits_{x\to +0}\left(1+x\right)^{\mathrm{ctg}x}$.

5.17. a) $\lim\limits_{x\to 5}\dfrac{20+x-x^2}{3x^2-11x-20}$; b) $\lim\limits_{x\to\infty}\dfrac{x+x^2+3x^3}{x^2+2x-3}$; c) $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{3x\,\mathrm{tg}3x}$; d) $\lim\limits_{x\to 1}\dfrac{\ln^2 x}{1-x}$.

5.18. a) $\lim\limits_{x\to 1}\dfrac{\sqrt{2-x}-1}{\sqrt{5-x}-2}$; b) $\lim\limits_{x\to\infty}\left(\dfrac{2x}{1+2x}\right)^{-4x}$; c) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{\left(\dfrac{\pi}{2}-x\right)^2}$; d) $\lim\limits_{x\to\frac{1}{2}+0}\ln 2x\ln(2x-1)$.

5.19. a) $\lim\limits_{x\to\infty}\dfrac{7x^5-3x^2+1}{5+2x^3+x^4+3x^5}$; b) $\lim\limits_{x\to -1}\dfrac{x^3+1}{x^3-x^2-2x}$; c) $\lim\limits_{x\to\infty}(x+2)(\ln(2x+1)-\ln(2x-1))$;

d) $\lim\limits_{x\to 0}\dfrac{1-e^{-x}-x}{\sin x+x}$.

5.20. a) $\lim\limits_{x\to 1}\dfrac{x^2-4x+3}{2x^2+x-3}$; b) $\lim\limits_{x\to\infty}\dfrac{4x^3+2x^2-3}{3x^4+x^3+2x^2+1}$; c) $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1-x^2}}{\cos 3x-\cos 5x}$; d) $\lim\limits_{x\to\infty}\left(x+3^x\right)^{\frac{1}{x}}$.

5.21. a) $\lim\limits_{x\to\infty}\dfrac{x^3+2x^2-5}{4x^2-x+1}$; b) $\lim\limits_{x\to 5}\dfrac{x^2-8x+15}{2x^2-9x-5}$; c) $\lim\limits_{x\to\infty}(x-4)(\ln(2-3x)-\ln(5-3x))$; d) $\lim\limits_{x\to 0}\dfrac{e^{3x}-3x-1}{\sin 2x-2x}$.

5.22. a) $\lim\limits_{x\to 2}\dfrac{\sqrt{2x}-x}{x^3-8}$; b) $\lim\limits_{x\to\infty}\left(\dfrac{5x-1}{5x+2}\right)^{3x}$; c) $\lim\limits_{x\to 0}\dfrac{5x^2}{\cos^3 2x-\cos 2x}$; d) $\lim\limits_{x\to 1}\dfrac{1-\sin^2\dfrac{\pi x}{2}}{(x-1)^2}$.

5.23. a) $\lim\limits_{x\to\frac{1}{2}}\dfrac{6x^2-5x+1}{8x^3-1}$; b) $\lim\limits_{x\to\infty}\left(\sqrt{4x^2-7x+4}-2x\right)$; c) $\lim\limits_{x\to 0}\left(1+\mathrm{tg}^2\sqrt{x}\right)^{\frac{3}{x}}$; d) $\lim\limits_{x\to+\infty}\dfrac{\pi-2\,\mathrm{arctg}\,x}{e^{\frac{3}{x}}-1}$.

5.24. a) $\lim\limits_{x\to\infty}\dfrac{3x^3+x}{x^4-3x^2+1}$; b) $\lim\limits_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$; c) $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{1-\cos 7x}$; d) $\lim\limits_{x\to+0}(\mathrm{ctg}\,x)^{\frac{1}{\ln x}}$.

5.25. a) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{\sqrt{x^2+1}-1}$; b) $\lim\limits_{x\to\infty}\dfrac{2x^4-x+7}{5x^4+x^2+11}$; c) $\lim\limits_{x\to\infty}(2x+5)(\ln(x-2)-\ln(x-1))$; d) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{e^x-e^{-x}-2x}$.

5.26. a) $\lim\limits_{x\to\infty}\dfrac{x^3-100x^2+1}{100x^2+15x}$; b) $\lim\limits_{x\to 1}\dfrac{x^3-x^2-x+1}{x^3+x-2}$; c) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{(1-\cos x)^2}}{\mathrm{tg}\,x}$; d) $\lim\limits_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$.

5.27. a) $\lim\limits_{x\to -3}\dfrac{1-\sqrt{4+x}}{5-\sqrt{22-x}}$; b) $\lim\limits_{x\to\infty}\dfrac{3x^4+9x^3-12}{4x^3+x^2-6}$; c) $\lim\limits_{x\to\infty}(3x+1)(\ln(1-3x)-\ln(5-3x))$; d) $\lim\limits_{x\to 0}\left(\dfrac{1}{x\sin x}-\dfrac{1}{x^2}\right)$.

5.28. a) $\lim\limits_{x\to\infty}\frac{(x+2)^2+(x-1)^2}{(x+1)^3-(x-1)^3}$; b) $\lim\limits_{x\to2}\frac{x^3+2x^2-8x}{1-\sqrt{3-x}}$; c) $\lim\limits_{x\to0}\frac{x\arcsin 3x}{\cos 2x-\cos 6x}$; d) $\lim\limits_{x\to+\infty}\left(\frac{\pi}{2}-\operatorname{arctg} x\right)^{\frac{1}{x}}$.

5.29. a) $\lim\limits_{x\to-1}\frac{\sqrt{2+x}+x}{x^3-4x^2-5x}$; b) $\lim\limits_{x\to\infty}\frac{2x^3+4x^2+x-3}{5x^2+2x-1}$; c) $\lim\limits_{x\to\infty}\left(\frac{x^2+4}{x^2-2}\right)^{x^2}$; d) $\lim\limits_{x\to1+0}\ln(x-1)\ln x$.

5.30. a) $\lim\limits_{x\to\infty}\frac{x^3+x-2}{7x^4+3x^3+x-1}$; b) $\lim\limits_{x\to3}\frac{2x^3-5x^2-9}{x^2+x-12}$; c) $\lim\limits_{x\to0}\frac{2\arcsin 3x}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$; d) $\lim\limits_{x\to+0}(\ln\operatorname{ctg} x)^{\operatorname{tg} x}$.

Task 6. **Investigate the next functions for continuity and establish the nature of discontinuity points, if any. In paragraph b, additionally build a graph of the function**

6.1. a) $f(x)=\frac{x+1}{x^2+2x}$; b) $f(x)=\begin{cases}1, & x<0,\\ 2^x, & 0<x\le 2,\\ x+3, & x>2.\end{cases}$

6.2. a) $f(x)=2^{\frac{1}{1-x}}$; b) $f(x)=\begin{cases}\sqrt{1-x}, & x\le 0,\\ 0, & 0<x\le 3,\\ x-3, & x>3.\end{cases}$

6.3. a) $f(x)=\frac{|x+4|}{x^2+4x}$; b) $f(x)=\begin{cases}2, & x<0,\\ \cos x+1, & 0\le x\le\pi,\\ 1-x, & x>\pi.\end{cases}$

6.4. a) $f(x)=\frac{\sin(x+2)}{x^2+x-2}$; b) $f(x)=\begin{cases}\ln x, & 0<x\le 1,\\ x-1, & 1<x\le 4,\\ x^2-10, & x>4.\end{cases}$

6.5. a) $f(x)=\dfrac{x^3+8}{x^2+2x}$; b) $f(x)=\begin{cases} x^2+1, & -\infty < x \le 0, \\ \mathrm{tg}\,x, & 0 < x \le \dfrac{\pi}{4}, \\ 1, & x > \dfrac{\pi}{4}. \end{cases}$

6.6. a) $f(x)=\dfrac{\ln(1+x)}{x^2-x}$; b) $f(x)=\begin{cases} \sqrt{1-x^2}, & -1 \le x \le 1, \\ x-1, & 1 < x \le 3, \\ -\sqrt{x}, & x > 3. \end{cases}$

6.7. a) $f(x)=\dfrac{2}{1+3^{1/x}}$; b) $f(x)=\begin{cases} \sin x, & -\infty < x \le \pi, \\ x-\pi, & \pi < x \le 2\pi, \\ \cos x, & x > 2\pi. \end{cases}$

6.8. a) $f(x)=\dfrac{x^2-2x+1}{x^4-x^3-x^2+x}$; b) $f(x)=\begin{cases} 1, & x \le 0, \\ 3^x, & 0 < x \le 1, \\ 2x+2, & x > 1. \end{cases}$

6.9. a) $f(x)=\dfrac{1-\cos x}{x^2}$; b) $f(x)=\begin{cases} -x, & x \le 0, \\ \ln x, & 0 < x \le e, \\ x-e, & x > e. \end{cases}$

6.10. a) $f(x)=\dfrac{2}{x^2-4}$; b) $f(x)=\begin{cases} -1, & x < 0, \\ \cos x, & 0 \le x \le \pi, \\ 1-x, & x > \pi. \end{cases}$

6.11. a) $f(x)=e^{\frac{1}{2x+4}}$; b) $f(x)=\begin{cases} 2\sqrt{x}, & 0 \le x < 1, \\ x^2+1, & 1 < x \le 2, \\ \dfrac{2}{x}+3, & x > 2. \end{cases}$

6.12. a) $f(x)=\dfrac{\arcsin x}{x^2-x}$; b) $f(x)=\begin{cases} x+1, & x\le 0,\\ 4^x, & 0<x\le 1,\\ 6-x^2, & x>1.\end{cases}$

6.13. a) $f(x)=\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$; b) $f(x)=\begin{cases} \sqrt{4-x^2}, & -2\le x\le 2,\\ x-2, & 2<x\le 4,\\ -2\sqrt{x}, & x>4.\end{cases}$

6.14. a) $f(x)=\dfrac{1-3^{\frac{1}{x}}}{1+3^{\frac{1}{x}}}$; b) $f(x)=\begin{cases} \operatorname{tg}x, & 0\le x\le \dfrac{\pi}{4},\\ \dfrac{2}{\pi}x, & \dfrac{\pi}{4}<x<\pi,\\ \sin x+2, & x\ge \pi.\end{cases}$

6.15. a) $f(x)=\operatorname{arctg}\dfrac{1}{x-1}$; b) $f(x)=\begin{cases} x^2+1, & -\infty<x\le 1,\\ \dfrac{2}{x}, & 1<x<4,\\ \dfrac{x-2}{4}, & x>4.\end{cases}$

6.16. a) $f(x)=\dfrac{\sqrt{x+8}-3}{x^2-1}$; b) $f(x)=\begin{cases} x^3+1, & x\le 0,\\ 3^x, & 0<x\le 2,\\ 6-x, & x>2.\end{cases}$

6.17. a) $f(x)=\dfrac{\sin\left(x^2-1\right)}{x^3-x}$; b) $f(x)=\begin{cases} x+1, & -\infty<x\le 3,\\ 3x-7, & 3<x\le 4,\\ 3+\sqrt{x}, & x>4.\end{cases}$

6.18. a) $f(x)=5^{\frac{1}{x-3}}$; b) $f(x)=\begin{cases} \cos x, & x<0,\\ 1-x, & 0<x\le 3,\\ x^2-5, & x>3.\end{cases}$

6.19. a) $f(x)=\dfrac{e^{x}-e^{-x}}{x}$; b) $f(x)=\begin{cases}1, & x\le 0,\\ \operatorname{tg}x, & 0<x<\dfrac{\pi}{2},\\ x-\dfrac{\pi}{2}, & x\ge\dfrac{\pi}{2}.\end{cases}$

6.20. a) $f(x)=\dfrac{\operatorname{tg}(x-1)}{x^{2}-3x+2}$; b) $f(x)=\begin{cases}x, & x\le 0,\\ x^{2}, & 0<x\le 1,\\ x^{2}+1, & x>1.\end{cases}$

6.21. a) $f(x)=\dfrac{\sqrt{7+x}-3}{x^{2}-4}$; b) $f(x)=\begin{cases}x+2, & x\le 1,\\ 3^{x}, & 1<x<2,\\ -x+5, & x\ge 2.\end{cases}$

6.22. a) $f(x)=\dfrac{\ln(1+2x)}{x^{2}-3x}$; b) $f(x)=\begin{cases}\sin x, & x\le 0,\\ x^{2}, & 0<x\le 1,\\ \sqrt{x}+1, & x>1.\end{cases}$

6.23. a) $f(x)=\dfrac{x^{2}-5x+6}{x^{2}-2x}$; b) $f(x)=\begin{cases}-x^{2}, & x\le 1,\\ \ln x, & 1<x\le e,\\ -3x+4, & x>e.\end{cases}$

6.24. a) $f(x)=\dfrac{1-\cos x}{3x^{2}-x^{3}}$; b) $f(x)=\begin{cases}x, & x\le 0,\\ -\sqrt{x}, & 0<x\le 4,\\ (x-4)^{2}, & x>4.\end{cases}$

6.25. a) $f(x)=\dfrac{1}{1+4^{\frac{1}{x-1}}}$; b) $f(x)=\begin{cases}1, & x\le 0,\\ \operatorname{ctg}x, & 0<x\le\dfrac{\pi}{2},\\ x-\dfrac{\pi}{2}, & x>\dfrac{\pi}{2}.\end{cases}$

6.26. a) $f(x)=\dfrac{|x-2|}{x^2-2x}$; b) $f(x)=\begin{cases}\sin x, & x\le 0,\\ x^2-3, & 0<x<2,\\ x-1, & x\ge 2.\end{cases}$

6.27. a) $f(x)=\dfrac{x^2+2x}{x^2\left(\sqrt{x+3}-1\right)}$; b) $f(x)=\begin{cases}-1, & x<0,\\ \operatorname{tg}x, & 0\le x\le \dfrac{\pi}{4},\\ \dfrac{4}{\pi}x, & x>\dfrac{\pi}{4}.\end{cases}$

6.28. a) $f(x)=\dfrac{\arcsin(x+1)}{x^2+x}$; b) $f(x)=\begin{cases}-x^2, & x\le 0,\\ \sqrt{x}, & 0<x\le 1,\\ \dfrac{3}{x}-1, & x>1.\end{cases}$

6.29. a) $f(x)=\dfrac{x^3-3x+2}{x-x^3}$; b) $f(x)=\begin{cases}-x, & x\le 0,\\ 1-x^2, & 0<x\le 1,\\ \ln x, & x>1.\end{cases}$

6.30. a) $f(x)=\dfrac{e^{2x}-1}{x^2+3x}$; b) $f(x)=\begin{cases}(x+2)^2, & x\le -2,\\ \sqrt{4-x^2}, & -2<x\le 0,\\ x, & x>0.\end{cases}$

Task 7. Find $\dfrac{dy}{dx}$ for the next functions

7.1. a) $y=\ln\left(\operatorname{tg}\dfrac{2x+1}{4}\right)$; b) $y=x^{\frac{1}{\ln x}}$;

c) $x^y-y^x=0$; d) $x=\arcsin t;\quad y=\sqrt{1-t^2}$.

7.2. a) $y=\sin^2\frac{x}{2}\operatorname{ctg}\frac{x}{2}$; b) $y=\left(\sqrt{x}\right)^{\sqrt[3]{x}}$;

c) $y^2\cos x=4\sin 3x$; d) $x=2\cos^2 t;\quad y=2\sin^2 t$.

7.3. a) $y=\sqrt{x}\arcsin\sqrt{x}+\sqrt{1-x}$; b) $y=x^{\arcsin x}$;

c) $x^4-6x^2y^2+9y^4-5x^2+15y^2-100=0$; d) $x=t^2+2;\quad y=\frac{1}{3}t^3-1$.

7.4. a) $y=\ln\left(3x^2+\sqrt{9x^4+1}\right)$; b) $y=x^{\ln x}$;

c) $\sin\left(y-x^2\right)-\ln\left(y-x^2\right)+2\sqrt{y-x^2}-3=0$; d) $x=\cos^3 t;\quad y=\sin t$.

7.5. a) $y=\operatorname{tg}\frac{x}{2}+\frac{2\cos x}{\sqrt{\cos 2x}}$; b) $y=(\ln x)^{\frac{1}{x}}$;

c) $xy^2-3\frac{y}{x}+2x^3=0$; d) $x=1+e^t;\quad y=t+e^{-t}$.

7.6. a) $y=\ln\sqrt{\frac{1+\sin x}{1-\sin x}}$; b) $y=\frac{(x+1)^8(x-3)^2}{\sqrt{(x+2)^5}}$;

c) $e^x+e^y-2^{xy}-3=0$; d) $x=5t^2;\quad y=4t^3+\operatorname{tg}t$.

7.7. a) $y=\operatorname{arctg}\sqrt{\frac{1-x}{1+x}}$; b) $y=(\sin x)^{\cos x}$;

c) $x^2\sin y+y^3\cos x-2x-3y+1=0$; d) $x=11\cos^2 t;\quad y=11\sin^3 t$.

7.8. a) $y=\arccos\sqrt{1-e^{x}}+\dfrac{5}{x^{4}}$; b) $y=(\sin x)^{\arcsin x}$;

c) $e^{y}+xy=3$; d) $x=e^{t}\cos t;\quad y=e^{t}\sin t$.

7.9. a) $y=\arcsin\dfrac{2x^{3}}{1+x^{6}}$; b) $y=x^{\sin x}$;

c) $\dfrac{y}{x}+e^{\frac{y}{x}}-\sqrt[3]{\dfrac{y}{x}}=0$; d) $x=2(t-\sin t);\quad y=2(1-\cos t)$.

7.10. a) $y=\ln\dfrac{\sqrt{x^{2}+1}-1}{\sqrt{x^{2}+1}+1}$; b) $y=x^{2}e^{x^{2}}\sin 2x$;

c) $x^{4}+y^{4}=x^{2}\cdot y^{2}$; d) $x=\arcsin t;\quad y=\ln\left(1-t^{2}\right)$.

7.11. a) $y=\log_{2}\left(\sin^{2}x\right)-3^{x^{2}}\sqrt{1+x}$; b) $y=(\sin x)^{\operatorname{tg}x}$;

c) $x\sin y+y\sin x=0$; d) $x=\ln\left(1+t^{2}\right);\quad y=t-\operatorname{arctg}t$.

7.12. a) $y=\arcsin\sqrt{\dfrac{\sin x}{1+\sin^{2}x}}$; b) $y=\left(x^{2}+3x-1\right)^{\frac{4}{x}}$;

c) $\dfrac{x^{2}}{25}+\dfrac{y^{2}}{9}=1$; d) $x=\ln t;\quad y=t^{2}-1$.

7.13. a) $y=\operatorname{arctg}(x+1)+\dfrac{x+1}{x^2+2x+2}$; b) $y=(x+1)^{\frac{3}{x}}$;

c) $2y\ln y=x$; d) $x=\arccos\sqrt{t}$; $y=\sqrt{t-t^2}$.

7.14. a) $y=\sqrt[5]{(x+4)^2}\arcsin 7x^2$; b) $y=\left(\sqrt{x}\right)^{\cos\sqrt{x}}$;

c) $\ln y-xy=5$; d) $x=t-\ln\sin t$; $y=t+\ln\cos t$.

7.15. a) $y=e^x-\sin e^x\cos^3 e^x$; b) $y=x^2e^{x^2}\ln x$;

c) $\sqrt{x}+\sqrt{y}=\sqrt{7xy}$; d) $x=4t\cos t$; $y=4t\sin t$.

7.16. a) $y=\ln\left(1-\dfrac{1}{x}\right)+\dfrac{1}{x}$; b) $y=\dfrac{(x-2)^2\sqrt[3]{x+1}}{(x-5)^3}$;

c) $xy=\operatorname{arctg}\dfrac{x}{y}$; d) $x=\operatorname{arctg}t$; $y=\ln\left(1+t^2\right)$.

7.17. a) $y=\dfrac{2+\operatorname{ctg}^3(2x-3)}{\ln\left(\sqrt{x}+2\right)}$; b) $y=(\operatorname{arctg}x)^{\ln x}$;

c) $y^2-5^{\cos^2 x}+\operatorname{tg}y=0$; d) $x=\operatorname{tg}t$; $y=\operatorname{ctg}t$.

7.18. a) $y=\ln\operatorname{tg}\dfrac{x}{2}+\cos x+\dfrac{1}{3}\cos^2 x$; b) $y=(\ln x)^x$;

c) $e^x\sin y-e^y\cos x=0$; d) $x=\dfrac{1}{\cos t}$; $y=\operatorname{tg}t$.

7.19. a) $y=\arccos\left(2e^{2x}-1\right)$; b) $y=\sqrt{x\sin x\sqrt{1-e^{x}}}$;

c) $\sin(xy)+\cos(xy)=0$; d) $x=2\sin t+\sin 2t;\quad y=2\cos t+\cos 2t$.

7.20. a) $y=\sqrt[3]{5x^{4}-2x-1}+e^{-\cos x}\sin 2x$; b) $y=(\arccos 3x)^{\lg(5x-1)}$;

c) $y=\cos(x+y)$; d) $x=\arcsin t;\quad y=\arccos t$.

7.21. a) $y=\ln\dfrac{\sqrt{x^{2}+2x}}{x+1}$; b) $y=\dfrac{(x+1)^{3}\sqrt[4]{x-2}}{\sqrt[3]{(x-3)^{2}}}$;

c) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}y$; d) $x=6\cos^{3}t;\quad y=6\sin^{3}t$.

7.22. a) $y=\ln\operatorname{tg}\dfrac{e^{2\sin x}}{4}$; b) $y=(\cos x)^{\frac{1}{x^{2}}}$;

c) $x-y=\arcsin x-\arcsin y$; d) $x=1+e^{4t};\quad y=4t+e^{-4t}$.

7.23. a) $y=\dfrac{e^{3x}}{\sqrt{3x^{2}-4x}}+\lg^{2}(1+\sin x)$; b) $y=\sqrt[9]{\dfrac{(x+3)\ln(2x-3)}{(x-3)^{2}}}$;

c) $e^{y}-e^{-y}-2xy=0$; d) $x=te^{t};\quad y=t^{2}+2t$.

7.24. a) $y=\operatorname{arctg}\dfrac{3x-x^{3}}{1-3x^{2}}$; b) $y=(\operatorname{ctg}x)^{\frac{1}{x}}$;

c) $2^{x}+2^{y}=2^{x+y}$; d) $x=t^{2}+2t;\quad y=\ln(t+1)$.

7.25. a) $y=\sqrt{2x+1}\left(\ln(2x+1)-2\right)$; b) $y=2\left(x^3-4\right)^{\sqrt{x}}$;

c) $\operatorname{arctg}\frac{y}{x}=\ln\sqrt{x^2+y^2}$; d) $x=2\cos t;\quad y=\sin 2t$.

7.26. a) $y=\sin^3 e^x\cos e^x-\frac{\ln x}{\sqrt{1-x^4}}$; b) $y=\frac{(x-7)^{10}\sqrt{3x-1}}{\sqrt[3]{(x+3)^5}}$;

c) $x+y=e^{\frac{x}{y}}$; d) $x=2t-t^2;\quad y=3t-t^3$.

7.27. a) $y=\frac{\operatorname{arctg}x}{2}-\ln\frac{x}{\sqrt{1+x^2}}$; b) $y=\left(\frac{x}{1+x^2}\right)^x$;

c) $x^2+y^2=25$; d) $x=\cos t+t\sin t;\quad y=\sin t-t\cos t$.

7.28. a) $y=e^x\sqrt{1-e^{2x}}-\arcsin e^x$; b) $y=\frac{\sqrt[3]{\left(x^2+3\right)\left(x^2-3\right)^2}}{(x+5)^4}$;

c) $\cos(xy)=x$; d) $x=e^{2t};\quad y=e^{3t}$.

7.29. a) $y=\sqrt[3]{\operatorname{arcctg}2x}+4^{-x}\ln^5(x+2)$; b) $y=\sqrt[3]{\frac{x\left(x^2+1\right)}{\left(x^2-1\right)^2}}$;

c) $x-y=e^{xy}$; d) $x=2\sin t;\quad y=4\cos^2 t$.

7.30. a) $y=-\frac{1+\ln\cos x}{\cos x}$; b) $y=\left(x^2+1\right)^{\sin x}$;

c) $y^3-3yx+6x^2=0$; d) $x=t^2+2t-3;\quad y=t+t^3$.

Task 8. **Explore the function given below and plot its graph**

8.1. $y=\frac{x^2-1}{x^2+1}$.

8.2. $y=\frac{x^2+1}{x}$.

8.3. $y=\frac{1-x^2}{x^2}$.

8.4. $y=\frac{x}{(1+x)^3}$.

8.5. $y=(x-1)e^x$.

8.6. $y=\frac{x^3}{1-x^2}$.

8.7. $y=\frac{4x^3}{1-x^3}$.

8.8. $y=\frac{x}{1+x^2}$.

8.9. $y=\frac{x^2}{x-1}$.

8.10. $y=\frac{4x^3+5}{x}$.

8.11. $y=x^2e^{-x}$.

8.12. $y=\frac{4x^2+1}{x}$.

8.13. $y=\frac{x^3}{x^2-1}$.

8.14. $y=x^3-6x^2+16$.

8.15. $y=\frac{x^2}{1-x}$.

8.16. $y=\frac{x^4}{1-x^2}$.

8.17. $y=\ln\left(4-x^2\right)$.

8.18. $y=\frac{x^4}{x^3-1}$.

8.19. $y=\frac{2-4x^2}{1-4x^2}$.

8.20. $y=x\sqrt{1-x^2}$.

8.21. $y=\frac{x^3}{2(1+x)^2}$.

8.22. $y=\frac{x^3+2}{2x}$.

8.23. $y=\frac{x}{x^2-4}$.

8.24. $y=\frac{x^3}{x^2-4}$.

8.25. $y=\frac{x^3}{(x-2)^2}$.

8.26. $y=x^3-6x^2+9x-3$.

8.27. $y=\frac{2+x^3}{x^2}$.

8.28. $y=\frac{x^4+1}{x^2}$.

8.29. $y=\ln\left(1-x^2\right)$.

8.30. $y=\frac{4x}{4+x^2}$.

Вспомогательный раздел

The Exam Questions

Educational discipline “Mathematics-1”

1. The matrix notations and terminology. The special types of matrices. Diagonal, triangular, and symmetric matrices. Describe the properties for such matrices. The zero matrices and the identity matrices.
2. Unary matrix operations: scalar multiplication of matrices, transpose for a matrix. Describe the properties for such operations.
3. The binary matrix operations: matrix addition, matrix subtraction. Describe the properties of the matrix addition and the scalar multiplication.
4. Multiplication of matrices. Properties of matrix multiplication.
5. Concept of a determinant. Evaluating determinants. Basic properties of determinants.
6. Minors and cofactors. Cofactor expansion. Determinant of a triangular matrix
7. Elementary row and column operations. Evaluating determinants by row reduction.
8. Inverse of a matrix. Theorem of inverse matrix. Properties of invertible matrices.
9. Methods for inverting matrices. Adjoint of a matrix. Inverse of a matrix using its adjoint. Calculation of inverse matrices by elementary transformations.
10. Elementary matrices and row operations. Equivalence theorem.
11. The matrix rank. Calculation of rank using minors, calculation of rank by elementary transformations.
12. The system of linear equations. Consistent and inconsistent systems. General solution.
13. Kronecker–Cappelli theorem. Number of solutions for a linear system. Solving linear systems by the matrix inversion. Cramer’s Rule.
14. Augmented matrices and elementary row operations. Echelon forms of matrices: row echelon and reduced row echelon form. Elimination methods: Gauss–Jordan elimination and Gaussian elimination.
15. Homogeneous linear systems. Free variables in homogeneous linear systems. Fundamental system of solutions.
16. The vectors in three-dimensional space. The linear vector operations. The components of a vector.

17. Rectangular orthogonal basis. Standard unit vectors. Decomposition of vectors into components.
18. Dot product. Properties of the dot product. A geometry problem solved using dot product. Applications of dot product. Calculating with dot products.
20. Cross product. Properties of cross product. Geometric interpretation of cross product. Determinant form of cross product.
21. Scalar triple product. Calculating a scalar triple product. Geometric interpretation. Properties of scalar triple product.
22. Equation of a line on a plane and in space.
23. A straight line in space. Equations of a straight line in space.
24. Equations of a plane in space. Relative position of lines and planes.
25. General equation of second-order curves in the Cartesian coordinate system. Circle, ellipses, hyperbolas, parabolas, finding the equations from some of its geometric properties.
26. Cylinders and quadric surfaces: cone, ellipsoid, hyperboloids, and paraboloids. Using traces to sketch the quadric surface. Applications of quadric surfaces.
27. Complex numbers, arithmetic operations on complex numbers, roots of complex numbers.
28. Limits of numerical sequences. Properties of convergent sequences. The number *e*. The natural logarithm.
29. Functions. Limits of functions. Limits at infinity. Elementary properties of limits. Infinite simal and infinite large functions.
30. Continuity of function at a point. Continuity of functions on an interval. Points of discontinuity and its classification.
31. Continuity of elementary functions. The most important limits. Properties of functions continuous on the closed interval.
32. The derivative of a function. The physical and geometrical concepts of the derivative.
33. Differentiation rules. Derivatives of composite functions: the chain rule.
34. Derivatives of basic elementary functions.
35. Derivatives of inverse functions, parametrically given function.
36. Implicit differentiation.
37. Logarithmic differentiation.
38. Higher derivatives. Higher-order differentials.
39. Rolle's theorem.
40. Lagrange’s theorem.
41. Cauchy's theorem.
42. L'Hôpital's rule.

43. Taylor’s formula. Lagrange’s form of the remainder, Peano form of the remainder, Cauchy form of the remainder.
44. The Taylor expansions of the main elementary functions and its applications.
45. Increasing functions and decreasing functions.
46. Local extreme values. Extreme values of functions.
47. The first derivative theorem for local extreme values.
48. The second derivative test for local extreme values.
49. Concavity. The second derivative test for concavity. Points of inflection.
50. Asymptotes. Strategy for graphing $y = f(x)$.
51. Vector-valued function. Derivatives of vector functions. Geometrical and mechanical meaning of a derivative.
52. The tangent line and the normal plane of a space curve.
53. Curvature of a plane curve. Radius of curvature.
54. The concept of the evolute and the involute.
55. The concept of the curvature and the torsion of a space curve.

The Exam Questions

Educational discipline “Mathematics-1”

1. The matrix notations and terminology. The special types of matrices. Diagonal, triangular, and symmetric matrices. Describe the properties for such matrices. The zero matrices and the identity matrices.
2. Unary matrix operations: scalar multiplication of matrices, transpose for a matrix. Describe the properties for such operations.
3. The binary matrix operations: matrix addition, matrix subtraction. Describe the properties of the matrix addition and the scalar multiplication.
4. Multiplication of matrices. Properties of matrix multiplication.
5. Concept of a determinant. Evaluating determinants. Basic properties of determinants.
6. Minors and cofactors. Cofactor expansion. Determinant of a triangular matrix
7. Elementary row and column operations. Evaluating determinants by row reduction.
8. Inverse of a matrix. Theorem of inverse matrix. Properties of invertible matrices.

9. Methods for inverting matrices. Adjoint of a matrix. Inverse of a matrix using its adjoint. Calculation of inverse matrices by elementary transformations.

10. Elementary matrices and row operations. Equivalence theorem.
11. The matrix rank. Calculation of rank using minors, calculation of rank by elementary transformations.
12. The system of linear equations. Consistent and inconsistent systems. General solution.
13. Kronecker–Cappelli theorem. Number of solutions for a linear system. Solving linear systems by the matrix inversion. Cramer’s Rule.

14. Augmented matrices and elementary row operations. Echelon forms of matrices: row echelon and reduced row echelon form. Elimination methods: Gauss–Jordan elimination and Gaussian elimination.

15. Homogeneous linear systems. Free variables in homogeneous linear systems. Fundamental system of solutions.
16. The vectors in three-dimensional space. The linear vector operations. The components of a vector.
17. Rectangular orthogonal basis. Standard unit vectors. Decomposition of vectors into components.
18. Dot product. Properties of the dot product. A geometry problem solved using dot product. Applications of dot product. Calculating with dot products.
20. Cross product. Properties of cross product. Geometric interpretation of cross product. Determinant form of cross product.
21. Scalar triple product. Calculating a scalar triple product. Geometric interpretation. Properties of scalar triple product.
22. Equation of a line on a plane and in space.
23. A straight line in space. Equations of a straight line in space.
24. Equations of a plane in space. Relative position of lines and planes.
25. General equation of second-order curves in the Cartesian coordinate system. Circle, ellipses, hyperbolas, parabolas, finding the equations from some of its geometric properties.
26. Cylinders and quadric surfaces: cone, ellipsoid, hyperboloids, and paraboloids. Using traces to sketch the quadric surface. Applications of quadric surfaces.
27. Complex numbers, arithmetic operations on complex numbers, roots of complex numbers.
28. Limits of numerical sequences. Properties of convergent sequences. The number *e*. The natural logarithm.
29. Functions. Limits of functions. Limits at infinity. Elementary properties of limits. Infinite simal and infinite large functions.
30. Continuity of function at a point. Continuity of functions on an interval. Points of discontinuity and its classification.

31. Continuity of elementary functions. The most important limits. Properties of functions continuous on the closed interval.
32. The derivative of a function. The physical and geometrical concepts of the derivative.
33. Differentiation rules. Derivatives of composite functions: the chain rule.
34. Derivatives of basic elementary functions.
35. Derivatives of inverse functions, parametrically given function.
36. Implicit differentiation.
37. Logarithmic differentiation.
38. Higher derivatives. Higher-order differentials.
39. Rolle's theorem.
40. Lagrange’s theorem.
41. Cauchy's theorem.
42. L'Hôpital's rule.
43. Taylor’s formula. Lagrange’s form of the remainder, Peano form of the remainder, Cauchy form of the remainder.
44. The Taylor expansions of the main elementary functions and its applications.
45. Increasing functions and decreasing functions.
46. Local extreme values. Extreme values of functions.
47. The first derivative theorem for local extreme values.
48. The second derivative test for local extreme values.
49. Concavity. The second derivative test for concavity. Points of inflection.
50. Asymptotes. Strategy for graphing .
51. Vector-valued function. Derivatives of vector functions. Geometrical and mechanical meaning of a derivative.
52. The tangent line and the normal plane of a space curve.
53. Curvature of a plane curve. Radius of curvature.
54. The concept of the evolute and the involute.
55. The concept of the curvature and the torsion of a space curve.

The list of books that are recommended for students

1. Calculus and analytic geometry. G. Thomas, R. Finnley, Addison-Wesley publishing company. 1998, 1281 pages.
2. Konspekt lectsii po vysshey mathematike. Polniy kurs. D. Pissmeniy. Airis-press, 2011. 606 pages. (in Russian)
3. Individualnie domashnie zadaniya po visshey mathematike. Part 1. Vysheyshaya shkola.1990, 271 pages. (in Russian)

BELARUSIAN NATIONAL TECHNICAL UNIVERSITY

EDUCATIONAL PLAN

APPROVE
Vice Rector for Academic work

___________Y. A. Nikolaychik
"___" ____________ 2023

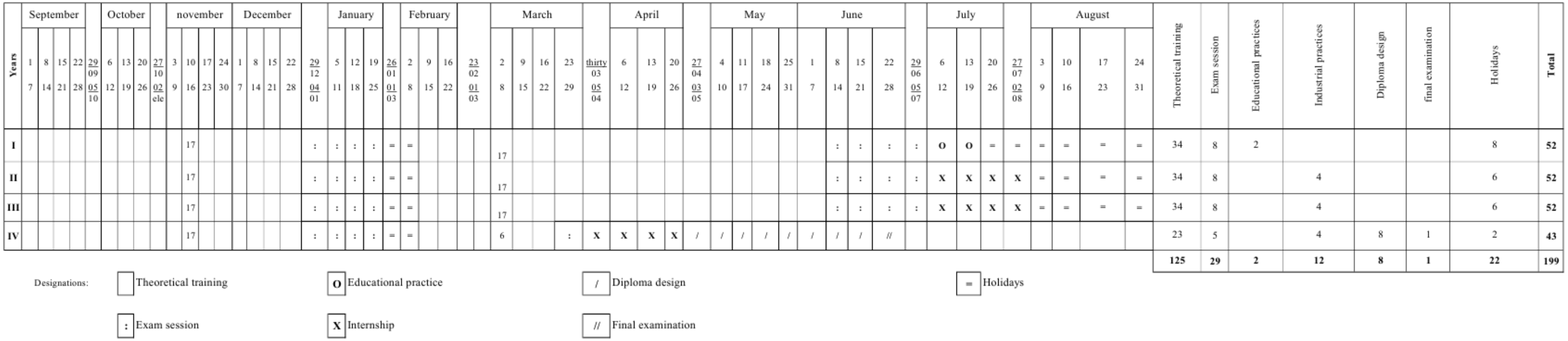
Specialty 6-05-0713-05 Robotic systems
Profiling: Industrial robots and robotic systems
Form of education: full-time

Qualification: Engineer
Degree: Bachelor
Duration of study: 4 years
Recruitment 2023

Registration number _____________ / account.

I.Schedule of the education process

II.Time budget summary (in weeks)

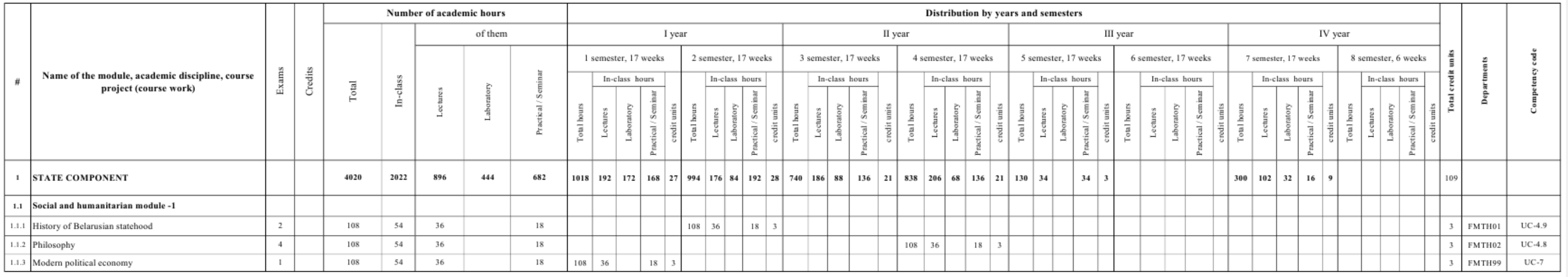


Years	September					October				november				December					January				February				March					April				May				June					July				August			
	1 7	8 14	15 21	22 28	29 09 05 10	6 12	13 19	20 26	27 10 02 ele	3 9	10 16	17 23	24 30	1 7	8 14	15 21	22 28	29 12 04 01	5 11	12 18	19 25	26 01 01 03	2 8	9 15	16 22	23 02 01 03	2 8	9 15	16 22	23 29	thirty 03 05 04	6 12	13 19	20 26	27 04 03 05	4 10	11 17	18 24	25 31	1 7	8 14	15 21	22 28	29 06 05 07	6 12	13 19	20 26	27 07 02 08	3 9	10 16	17 23	24 31
I											17							:	:	:	:	=	=				17														:	:	:	:	O	O	=	=	=	=	=	=
II											17							:	:	:	:	=	=				17														:	:	:	:	X	X	X	X	=	=	=	=
III											17							:	:	:	:	=	=				17														:	:	:	:	X	X	X	X	=	=	=	=
IV											17							:	:	:	:	=	=				6			:	X	X	X	X	/	/	/	/	/	/	/	/	//									

Theoretical training	Exam session	Educational practices	Industrial practices	Diploma design	final examination	Holidays	**Total**
34	8	2				8	**52**
34	8		4			6	**52**
34	8		4			6	**52**
23	5		4	8	1	2	**43**
125	**29**	**2**	**12**	**8**	**1**	**22**	**199**

Designations:
☐ Theoretical training
O Educational practice
/ Diploma design
= Holidays
: Exam session
X Internship
// Final examination

III.Plan of the education process



#	Name of the module, academic discipline, course project (course work)	Exams	Credits	Number of academic hours					Distribution by years and semesters																																							Total credit units	Departments	Competency code
				Total	In-class	of them			I year										II year										III year										IV year												
						Lectures	Laboratory	Practical / Seminar	1 semester, 17 weeks					2 semester, 17 weeks					3 semester, 17 weeks					4 semester, 17 weeks					5 semester, 17 weeks					6 semester, 17 weeks					7 semester, 17 weeks					8 semester, 6 weeks							
										In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours						
									Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units			
1	**STATE COMPONENT**			**4020**	**2022**	**896**	**444**	**682**	**1018**	**192**	**172**	**168**	**27**	**994**	**176**	**84**	**192**	**28**	**740**	**186**	**88**	**136**	**21**	**838**	**206**	**68**	**136**	**21**	**130**	**34**		**34**	**3**						**300**	**102**	**32**	**16**	**9**						109		
1.1	**Social and humanitarian module -1**																																																		
1.1.1	History of Belarusian statehood	2		108	54	36		18						108	36		18	3																															3	FMTH01	UC-4.9
1.1.2	Philosophy	4		108	54	36		18																108	36		18	3																					3	FMTH02	UC-4.8
1.1.3	Modern political economy	1		108	54	36		18	108	36		18	3																																				3	FMTH99	UC-7

#	Name of the module, academic discipline, course project (course work)	Exams	Credits	Number of academic hours					Distribution by years and semesters																																								Total credit units	Departments	Competency code
				Total	In-class	of them			I year										II year										III year										IV year												
									1 semester, 17 weeks					2 semester, 17 weeks					3 semester, 17 weeks					4 semester, 17 weeks					5 semester, 17 weeks					6 semester, 17 weeks					7 semester, 17 weeks					8 semester, 6 weeks							
						Lectures	Laboratory	Practical / Seminar	Total hours	In-class hours			credit units	Total hours	In-class hours			credit units	Total hours	In-class hours			credit units	Total hours	In-class hours			credit units	Total hours	In-class hours			credit units	Total hours	In-class hours			credit units	Total hours	In-class hours			credit units	Total hours	In-class hours			credit units			
										Lectures	Laboratory	Practical / Seminar			Lectures	Laboratory	Practical / Seminar			Lectures	Laboratory	Practical / Seminar			Lectures	Laboratory	Practical / Seminar			Lectures	Laboratory	Practical / Seminar			Lectures	Laboratory	Practical / Seminar			Lectures	Laboratory	Practical / Seminar			Lectures	Laboratory	Practical / Seminar				
1.2	**Science module**																																																		
1.2.1	Physics	1,2		390	204	104	68	32	190	52	34	16	5	200	52	34	16	5																															10	FITR47	BPC-1
1.2.2	Higher mathematics	1,2,3,4		566	306	136		170	130	34		34	3	176	34		68	5	130	34		34	3	130	34		34	3																					14	FITR48	BPC-2
1.3	**Linguistic module**																																																		
1.3.1	Foreign language	2	1	200	100			100	100			50	3	100			50	3																															6	FMTH09	UC-3
1.3.2	Belarusian language (professional vocabulary)		3	90	34			34											90			34	3																										3	FMTH88	UC-11
1.4	**Module "Ecology and safety"**																																																		
1.4.1	Fundamentals of environmental and energy sustainability of production		7	100	50	34	16																																100	34	16		3						3	FMEE84 FMTH92	UC-6, BPC-4
1.4.2	Protection of the population and objects from emergency situations. Radiation safety		7	100	50	34	16																																100	34	16		3						3	FITR47	BPC-5
1.4.3	Occupational Safety and Health	7		100	50	34		16																															100	34		16	3						3	MTF35	BPC-6
1.5	**Module "Fundamentals of design"**																																																		
1.5.1	Engineering graphics	1	2[1]	200	100	36		64	100	18		32	3	100	18		32	3																															6	ATF11	BPC-7
1.5.2	Rationing of accuracy and technical measurements		2	100	42	18	16	8						100	18	16	8	3																															3	FI81	BPC-8
	Coursework on the academic discipline "Regulation of accuracy and technical measurements"			40										40				1																															1		
1.6	**Module "Theoretical Mechanics"**																																																		
1.6.1	Theoretical mechanics	4		130	68	34		34																130	34		34	3																					3	MSF25	BPC-9
1.6.2	Theory of mechanisms and machines	3		130	84	50	18	16											130	50	18	16	4																										4	MSF26	BPC-10
1.7	**Module "Applied mechanics"**																																																		
1.7.1	Details and mechanisms of devices and machines	3	4	260	140	68	36	36											130	34	18	18	4	130	34	18	18	3																					7	MSF26	BPC-11
	Course work on the academic discipline "Details and mechanisms of devices and machines"			40																				40				1																					1		
1.7.2	Materials Science	3		130	68	34	18	16											130	34	18	16	3																										3	MTF34	BPC-12
1.7.3	Strength of materials	5		130	68	34		34																					130	34		34	3																3	MSF22	BPC-13
1.8	**Module "Informatics and Computer Engineering"**																																																		
1.8.1	Computer science	1		130	70	18	52		130	18	52		4																																				4	FITR45	UC-1,2, BPC-14
1.8.2	Architecture and system software of computers		1	130	68	16	52		130	16	52		3																																				3	FITR45	BPC-15
1.8.3	Programming languages and technologies		1,2	260	122	36	68	18	130	18	34	18	3	130	18	34		4																															7	FITR45	UC-5, BPC-16
	Coursework on the academic discipline "Programming Languages and Technologies"			40										40				1																															1		
1.9	**Module "Electrical engineering and electronics"**																																																		UC-5
1.9.1	Theoretical Foundations of Electronics	4		130	66	34	16	16																130	34	16	16	3																					3	PEF39	BPC-17
1.9.2	Electronics and circuitry	3	4	260	170	68	68	34											130	34	34	18	4	130	34	34	16	4																					8	FITR45	BPC-18
	Coursework in the academic discipline "Electronics and circuitry"			40																				40				1																					1		

#	Name of the module, academic discipline, course project (course work)	Exams	Credits	Number of academic hours					Distribution by years and semesters																																							Total credit units	Departments	Competency code
				Total	In-class	of them			I year										II year										III year										IV year												
									1 semester, 17 weeks					2 semester, 17 weeks					3 semester, 17 weeks					4 semester, 17 weeks					5 semester, 17 weeks					6 semester, 17 weeks					7 semester, 17 weeks					8 semester, 6 weeks							
										In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours						
						Lectures	Laboratory	Practical / Seminar	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units			
2	**COMPONENT OF EDUCATIONAL INSTITUTION**			**2732**	**1356**	**516**	**478**	**362**						**72**	**18**		**18**	**2**	**250**	**18**	**68**	**18**	**6**	**210**	**36**	**34**	**52**	**6**	**948**	**188**	**138**	**138**	**23**	**1018**	**206**	**204**	**100**	**28**	**144**	**34**	**34**	**18**	**4**	**90**	**16**		**18**	**3**	72		
2.1	**Social and humanitarian module - 2**																																																		
2.1.1	Political science		3[1]	72	36	18		18											72	18		18	2																										2	FMTH04	UC-4,5,6,14,15
2.1.2	Fundamentals of psychology and pedagogy		4[1]	72	36	18		18																72	18		18	2																					2	EPF10	UC-10,16,17
2.1.3	History of science and technology / The Great Patriotic War of the Soviet people (in the context of World War II)		2[1]	72	36	18		18						72	18		18	2																															2	FMTH01	UC-18.12
2.2	**Module "Automation of engineering calculations and design"**																																																		
2.2.1	Math software packages		6	100	52	18	34																											100	18	34		3											3	FITR45	SC-1
2.2.2	Computer-aided design		3	138	68		68												138		68		3																										3	FITR45	SC-2
	Coursework in the academic discipline "Computer-aided design"			40															40				1																										1		
2.3	**Module "Technology and equipment"**																																																		
2.3.1	Technology and equipment of robotic production	6	5	276	172	68	36	68																					138	34	18	34	4	138	34	18	34	4											8	FITR45	SC-3
	Course project on the discipline "Technology and equipment of robotic production"			60																														60				2											2		
2.4	**Module "Enterprise economy"**																																																		
2.4.1	Enterprise economy		5	130	68	34		34																					130	34		34	3																3	MSF28	SC-4
	Coursework on the academic discipline "Enterprise economy"			40																									40				1																1		
2.4.2	Fundamentals of Intellectual Property Management [2]		8	90	34	16		18																																				90	16		18	3	3	MSF28	SC-5
2.5	**Module "Executive system of industrial robots"**																																																		
2.5.1	Mechanics of industrial robots	5		130	68	34	16	18																					130	34	16	18	3																3	FITR45	SC-6
2.5.2	Electric drive of industrial robots	5	6	340	170	68	68	34																					170	34	34	18	4	170	34	34	16	4											8	FITR45	SC-7
	Course project on the discipline "Electric drive of robotic systems"			60																														60				2											2		
2.5.3	Hydraulic and pneumatic drive	6		100	50	34	16																											100	34	16		3											3	ATF19	SC-8
2.6	**Module "Control system of industrial robots"**																																																		
2.6.1	Theory of automatic control	5	6	260	170	68	68	34																					130	34	34	18	3	130	34	34	16	4											7	FITR45	SC-9
	Coursework on the academic discipline "Theory of automatic control"			40																									40				1																1		
2.6.2	Control systems for industrial robots	6		130	86	34	34	18																										130	34	34	18	3											3	FITR45	UC-1, SC-10
2.6.3	Information and measuring systems of robots	7		144	86	34	34	18																															144	34	34	18	4						4	FITR45	UC-1, SC-11

#	Name of the module, academic discipline, course project (course work)	Exams	Credits	Number of academic hours					Distribution by years and semesters																																								Total credit units	Departments	Competency code
						of them			I year										II year										III year										IV year												
									1 semester, 17 weeks					2 semester, 17 weeks					3 semester, 17 weeks					4 semester, 17 weeks					5 semester, 17 weeks					6 semester, 17 weeks					7 semester, 17 weeks					8 semester, 6 weeks							
										In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours						
				Total	In-class	Lectures	Laboratory	Practical / Seminar	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units			
2.7	**Module "Software for industrial robots"**																																																		
2.7.1	Languages and systems for programming industrial robots		6	130	68	18	34	16																										130	18	34	16	3											3	FITR45	SC-12
2.7.2	Mathematical and system software for industrial robots	4	5	268	156	36	70	50																138	18	34	34	4	130	18	36	16	3																7	FITR45	SC-13
	Coursework on the academic discipline "Mathematical and software of industrial robots"			40																									40				1																1		
	Optional modules																																																		
2.8	**Module "Industrial robots and robotic complexes in mechanical engineering"**			**956**	**402**	**140**	**140**	**122**																															**696**	**104**	**104**	**86**	**18**	**260**	**36**	**36**	**36**	**8**	26		
2.8.1	Network technologies	7		138	86	34	34	18																															138	34	34	18	3						3	FITR45	SC-14
2.8.2	RTC control systems in mechanical engineering	7,8		300	156	52	52	52																															170	34	34	34	5	130	18	18	18	4	9	FITR45	UC-5, SC-15
	Coursework on the academic discipline "RTC control systems in mechanical engineering"			40																																			40				1						1		
2.8.3	Program control of RTC in mechanical engineering	7,8		268	106	36	36	34																															138	18	18	16	4	130	18	18	18	4	8	FITR45	UC-5, SC-16
	Course work on the academic discipline "Program control of the RTC in mechanical engineering"			40																																			40				1						1		
2.8.4	Modeling and design of RTC in mechanical engineering		7	130	54	18	18	18																															130	18	18	18	3						3	FITR45	UC-5, SC-17
	Coursework on the academic discipline "Modeling and design of RTC in mechanical engineering"			40																																			40				1						1		
2.9	**Module "Industrial robots and robotic systems in instrumentation"**			**956**	**402**	**140**	**140**	**122**																															**696**	**104**	**104**	**86**	**18**	**260**	**36**	**36**	**36**	**8**	26		
2.9.1	Network technologies	7		138	86	34	34	18																															138	34	34	18	3						3	FITR45	SC-14
2.9.2	RTC control systems in instrumentation	7,8		300	156	52	52	52																															170	34	34	34	5	130	18	18	18	4	9	FITR45	UC-5, SC-15
	Coursework on the academic discipline "RTC control systems in instrumentation			40																																			40				1						1		
2.9.3	Program control of RTC in instrumentation	7,8		268	106	36	36	34																															138	18	18	16	4	130	18	18	18	4	8	FITR45	UC-5, SC-16
	Course work on the academic discipline "Program control of the RTC in instrumentation"			40																																			40				1						1		
2.9.4	Modeling and design of RTC in instrumentation		7	130	54	18	18	18																															130	18	18	18	3						3	FITR45	UC-5, SC-17
	Course work on the academic discipline "Modeling and design of RTC in instrumentation"			40																																			40				1						1		

#	Name of the module, academic discipline, course project (course work)	Exams	Credits	Number of academic hours					Distribution by years and semesters																																								Total credit units	Departments	Competency code
				Total	In-class	of them			I year										II year										III year										IV year												
						Lectures	Laboratory	Practical / Seminar	1 semester, 17 weeks					2 semester, 17 weeks					3 semester, 17 weeks					4 semester, 17 weeks					5 semester, 17 weeks					6 semester, 17 weeks					7 semester, 17 weeks					8 semester, 6 weeks							
										In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours					In-class hours							
									Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units	Total hours	Lectures	Laboratory	Practical / Seminar	credit units			
2.10	**OPTIONAL DISCIPLINES**																																																		
2.10.1	Corruption and its public danger			/10	/10	/10								/10			/10																																	FMTH99	UC-19
2.10.2	Introduction to the specialty			/16	/16	/16			/16			/16																																						FITR45	SC-18
2.10.3	Physical Culture			/68	/68			/68																					/34			/34		/34			/34														STF113	UC-13
2.11	**ADDITIONAL TYPES OF TRAINING**																																																		
2.11.1	Physical Culture		/1-6	/340	/340	/4		/336	/68	/4		/64		/68			/68		/68			/68		/68			/68		/34			/34		/34			/34														STF113	UC-13
2.11.2	Military training**			240/360																																													MTF	UC-5

**Military training curricula for the first level with a volume of 240 hours, for the second level - with a volume of 360 hours are implemented by the military-technical faculty of the Belarusian National Technical University. This discipline is not mandatory for all students.

Number of hours of training sessions total				7708	3780	1552	1062	1166	1018	192	172	168	27	1066	194	84	210	30	990	204	156	154	27	1048	242	102	188	27	1078	222	138	172	26	1018	206	204	100	28	1140	240	170	120	31	350	52	36	54	11	207		
Number of hours of study per week									31					29					30					31					31					30					31					24							
Number of course projects				2																														2																	
Number of course works				11										2					1					2					3										3												
Number of exams				31					5					4					4					4					4					3					5					2							
Number of credits				24					3					4					3					3					3					4					3					1							

IV. Educational practices				V. Industrial Practices				VI. Diploma design			VII. final examination
Name of practice	Semester	Weeks	Credit units	Name of practice	Semester	Weeks	Credit units	Semester	Weeks	Credit units	
Introductory	2	2	3	Technological	4	4	6	8	8	12	Graduation project defense
				Design and technological	6	4	6				
				Undergraduate	8	4	6				

VIII.Competence matrix

Competency code	Name of competence	Module code, academic discipline
UC-1	Own the basics of research activities, search, analyze and synthesize information	1.8.1, 2.6.2, 2.6.3
UC-2	Solve standard tasks of professional activity based on the use of information and communication technologies	1.8.1
UC-3	Communicate in a foreign language to solve problems of interpersonal and intercultural interaction	1.3.1
UC-4	Work in a team, tolerantly perceive social, ethnic, confessional, cultural and other differences	1.1.1, 1.1.3, 2.1.1
UC-5	Be capable of self-development and improvement in professional activities	1.8.3, 1.9, 2.1.1, 2.8.2, 2.8.3, 2.8.4, 2.9.2, 2.9.3, 2.9.4
UC-6	Show initiative and adapt to changes in professional activity	1.4.1, 2.1.1
UC-7	Have the ability to analyze the economic system of society in its dynamics, the laws of its functioning and development in order to understand the factors of emergence and development directions of socio-economic systems, their ability to meet the needs of people, identify factors and mechanisms of political and socio-economic processes, use tools of economic analysis to assess the political process of economic decision-making and the effectiveness of economic policy	1.1.2
UC-8	Possess a modern culture of thinking, a humanistic worldview, an analytical and innovative-critical style of cognitive, socio-practical and communicative activity, use the basics of philosophical knowledge in professional activities, independently assimilate philosophical knowledge and build a worldview position on their basis	1.1.3
UC-9	To have the ability to analyze the processes of state building in different historical periods, to identify factors and mechanisms of historical changes, to determine the socio-political significance of historical events (personalities, artifacts and symbols) for modern Belarusian statehood, to perfectly use the identified patterns in the process of forming civic identity	1.1.1
UC-10	Have the ability to analyze political events, processes, relationships, use the culture of political thinking and behavior, use the basics of political science knowledge to form a culture of conscious and rational political choice, affirm socially oriented values	2.1.2
UC-11	Use the basic concepts and terms of the special vocabulary of the Belarusian language in professional activities	1.3.2
UC-12	To have the ability to formulate their own worldview principles at the level of the feat of the Belarusian people and the historical lessons of the Great Patriotic War, to preserve and increase the historical memory of the role of the Soviet Union and its peoples in the Victory over German Nazism, to transmit to new generations the historical truth and norms of behavior, values and traditions, developed by the Belarusian people during the period of overcoming the tragic events of the Great Patriotic War	2.1.3
UC-13	Use the means of physical culture and sports to maintain and promote health, prevent diseases	2.10.3, 2.11.1
UC-14	Analyze the theoretical and methodological foundations of the problem of the professional development of a person in the labor process	2.1.1
UC-15	Know the specifics and patterns of development of world cultures	2.1.1
UC-16	Analyze various aspects of modern political institutions, determine the characteristics and types of political systems	2.1.2
UC-17	Use forms, techniques, methods and laws of intellectual cognitive activity in the professional field	2.1.2
UC-18	Evaluate the main events and stages in history to form a holistic view of the development of science and technology	2.1.3
UC-19	Possess the qualities of citizenship, understand the social danger of corruption	2.10.1
BPC-1	Use the basic laws of physics to solve applied engineering problems, use measuring instruments to analyze physical phenomena and processes	1.2.1
BPC-2	Apply the analysis of methods for using the mathematical apparatus of algebra, analytical geometry, differential and integral functions to solve applied engineering problems	1.2.2
BPC-3	Apply the mathematical apparatus of differential equations, series, integral functions of several variables to solve applied engineering problems	1.2.3
BPC-4	Use the basics of environmental and energy sustainability of production for their application in professional activities	1.4.1
BPC-5	Apply basic methods of protection and behavior in emergency situations and radiation hazards	1.4.2
BPC-6	Apply basic health and safety rules to prevent injury in the workplace	1.4.3
BPC-7	Use the skills of reading and performing graphic materials and technical documentation when designing devices using ESKD standards to unify technical documentation	1.5.1
BPC-8	Use knowledge about the basics of standardizing the accuracy and quality of products, apply practical skills in using measuring tools and devices when designing devices	1.5.2
BPC-9	Use the provisions of theoretical mechanics for the calculation of mechanical systems in the design of devices	1.6.1
BPC-10	Apply methods of analysis and synthesis of mechanisms to derive a mathematical description of technical systems	1.6.2
BPC-11	Calculate and develop the design of mechanisms and devices, taking into account the classification of the main parts and mechanisms of machines and devices for the unification of production processes	1.7.1
BPC-12	Make a choice of basic structural and electrical materials in the design of mechanical structures	1.7.2
BPC-13	Calculate parts and structures for strength, rigidity and stability when designing devices	1.7.3
BPC-14	Search, store and analyze information from various sources when compiling technical documentation, use the main methods of presenting information in the required format using information, computer and network technologies	1.8.1
BPC-15	Use knowledge about the device (composition) and the principle of operation of the hardware and system software of the computer when choosing the electrical components of devices, complete (upgrade) the computer, install its software	1.8.2
BPC-16	Use software and knowledge about one of the universal algorithmic programming languages. when creating the software of the designed device	1.8.3
BPC-17	Use the basic principles and methods for calculating the characteristics of electrical circuits and electromagnetic fields for their use in production	1.9.1
BPC-18	Make a choice of the element base of electronic components when designing electrical systems of the designed device, use the skills of reading and developing electrical circuits when drawing up technical documentation	1.9.2

Competency code	Name of competence	Module code, academic discipline
SC-1	Use one of the universal packages of computer mathematics, one of specialized packages for statistical analysis, processing of tabular (matrix) information	2.2.1
SC-2	Use the general principles of computer-aided design when creating devices, create various elements of the designed device in one of the modern systems of three-dimensional solid-state graphic modeling and design (CAD)	2.2.2
SC-3	Use methods for designing automated and robotic technological processes, analyze and select technological equipment	2.3
SC-4	Use the basics of enterprise economics, methods for conducting a survey of production and registration of a feasibility study of the project when developing production	2.4.1
SC-5	Use standard methodologies for registering patent inventions and controlling copyright compliance	2.4.2
SC-6	Apply the designs of manipulators and working bodies of industrial robots in the design and calculation of the characteristics of the mechanical system of the robot	2.5.1
SC-7	Calculate the required characteristics of the electric drive of industrial robots and select technical means for creating an electric drive control system, develop electric drive control schemes	2.5.2
SC-8	Calculate the required characteristics of pneumatic and hydraulic drives of industrial robots, select technical means for creating a control system for pneumatic and hydraulic drives, develop control schemes for pneumatic and hydraulic drives	2.5.3
SC-9	Use the methods of mathematical description of automatic control systems (ACS) when designing robots, use the skills of working in the batch modeling software for ACS MatLab Simulinc	2.6.1
SC-10	Use knowledge about the classification of modern representatives of the cyclic, positional and contour systems of industrial robots during their operation, diagnosis and repair	2.6.2
SC-11	Use knowledge of the main types of sensors and electrical circuits for their connection to the control system of industrial robots, methods of calculation and selection of technical means of the information-measuring subsystem of industrial robots when designing robotic devices	2.6.3
SC-12	Use one of the modern languages and systems for programming industrial robots when creating software, use methods for developing a control program when compiling it	2.7.1
SC-13	Use the methods of mathematical apparatus to describe the position and trajectory of manipulators, use the system software of industrial robots when working in production	2.7.2
SC-14	Design and operate industrial local area networks, taking into account the hardware and software of global and local computer networks	2.8.1, 2.9.1
SC-15	Operate, diagnose, repair RTC control systems, create diagrams of RTC control systems in production	2.8.2, 2.9.2
SC-16	Use the basics of syntax and algorithmization in the ISO-7bit CNC programming language when creating a control program, create control programs in one of the PLC and microcontroller programming languages	2.8.3, 2.9.3
SC-17	Use modern software systems for modeling production systems (Plant Simulation, AutoMOD or similar), apply simulation results to justify RTC projects	2.8.4, 2.9.4
SC-18	Know the main provisions, state and global trends in the development of robotics	2.10.2

Developed on the basis of the educational standard in the specialty 6-05-0713-05 "Robotic systems" and an exemplary curriculum (registration No. 6-05-07-049 / pr. from 02/15/2023).

1 Differentiated credit.

2 When drawing up the curriculum of an educational institution in the specialty, the academic discipline "Fundamentals of Intellectual Property Management" is planned as a discipline of a component of an educational institution or a discipline of choice.

Head of CEEDaOEP

___________________________T.N. Kanashevich
"_____" ___________________________2023

Head of the Department for Long-term Planning of Educational Activities of the CEEDaOEP

___________________________I.M. Kovaleva
"_____" ___________________________2023

Dean of FITR

___________________________A.M. Avsievich
"_____" ___________________________2023

Head of the Department "Robotic systems"

___________________________A.R. Okolov
"_____" ___________________________2023

Developer

_____________________S.S.Prokhorovich
"_____" ___________________________2023

Responsible for the formation of lecture streams

_______________________A.V. Manyukevich
"_____" ___________________________2023

Comptroller

____
"_____" ___________________________2023

Recommended for approval by the Scientific and Methodological Council of BNTU
Protocol # ____ dated _________ 2023