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МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ЭКОНОМИЧЕСКИХ СИСТЕМ И БИЗНЕС-ПРОЦЕССОВ С КОПУЛАМИ

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Аннотация. Известно, что многомерное распределение Пуассона, экспоненциальное или нормальное распределение, можно рассматривать как хорошую модель для оценки совместного распределения множества микроэкономических, макроэкономических и финансовых переменных. Но иногда эти распределения могут не дать хороших результатов. Это приводит к проблеме поиска многомерных моделей, которые лучше всего соответствуют этому распределению. Одним из возможных способов решения этой проблемы является использование теории функций, известных как копулы. В данной статье рассматриваются проблемы математического моделирования экономических систем и бизнес-процессов с использованием метода копулы.

Ключевые слова: математическое моделирование, метод копулы, экономические системы, бизнес-процессы, совместное распределение.

MATHEMATICAL MODELING OF ECONOMIC SYSTEMS AND BUSINESS PROCESSES WITH COPULAS

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It is known that the multivariate Poisson distribution, exponential distribution or normal distribution, can be considered as a good model for estimating the joint distribution of abounding microeconomic, macroeconomic and financial variables. But sometimes these distributions may not

give good results. This leads to the problem of finding multivariate models that best fit this distribution. One possible way to solve this problem is to use the theory of functions known as copulas. This article discusses the problems of mathematical modeling of economic systems and business processes using the copula method.
Keywords: mathematical modeling, copula method, economic systems, business processes, joint distribution.

Numerous scientific and practical researches conducted all over the world are mainly focused on the use of mathematical modeling methods. In particular, statistical models are mathematical models in various fields, such as economics, and business processes, physics, astronomy, biology, medicine, demography, sociology, psychology, engineering, mechanical engineering, computer science, natural sciences, and other related fields.

Currently, there is a need to study statistical models of multidimensional random quantities and their distributions, and such problems are in the focus of attention of specialists. Nonparametric estimates of the multivariate reliability function for complete and incomplete samples are important in many practical problems. This is the basis for determining the adequacy of the models. In fact, the process of nonparametric statistical estimation of distributions is a more complicated process than parametric estimation. It is known that the multivariate Poisson distribution, exponential distribution or normal distribution, can be considered as a good model for estimating the joint distribution of abounding microeconomic, macroeconomic and financial variables. But sometimes these distributions may not give good results. This leads to the problem of finding multivariate models that best fit this distribution. One possible way to solve this problem is to use the theory of functions known as copulas. The word copula means link, connection. The scientific work of mathematicians Hoeffding (1940) and Sklar (1959) played an important role in the initial formation of the theory of these functions. By the end of the 1990s, the attention of specialists was attracted by mathematical modeling using copula functions and their practical application. The copula function method consists of forming a correlation structure by connecting the marginal distributions to the joint distribution using another auxiliary link function (see, [1], [8]-[10]).

The copula methodology separates marginal distributions from the dependence structure, linking them to the joint distribution through the copula function. Professor of Illinois University of Technology of USA Abe Sklar introduced the theory of copula in 1959. Early monographs on copula functions are Joe (1997) (focusing on new probabilistic concepts around copulas) and Nelsen (1999, 2006) (a popular, readable introduction to the copula). An interesting historical perspective and introduction to copulas can be found in Durante and Sempi (2010). A more advanced probabilistic treatment of copulas is Durante and Sempi (2015). Currently, copula functions have been successfully applied in finance-insurance (Cherubini, Umberto, et al., 2013), biostatistics (Lambert, Vandenhende, 2002), hydrology (Zhang, Singh, 2006), and climate science (Salvadori, De Michel, 2007) is being used (see, [1], [5], [10]).

Statistically, a copula can be defined as follows.

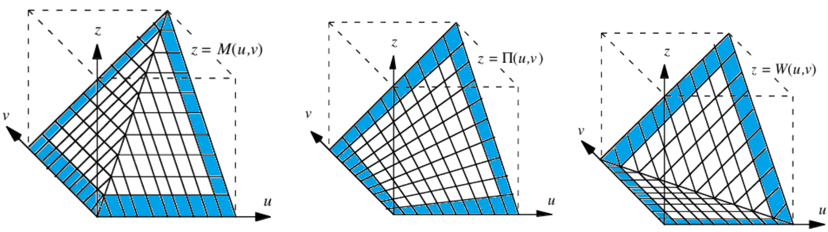
Definition. A copula is a bivariate distribution function with uniform marginals.

Examples ([5]-[8]). a) (Independence Copula). The function

$\Pi(u,v):[0,1]^2 \to [0,1]$ given by

$\Pi(u,v) = u \cdot v, \quad u,v \in [0,1],$

is called the independence copula (see, picture 1a).



Picture 1 – Visual graphs of. a) Independent copula; b) comonotone; c)countermonotonic copulas.

b) (Comonotonicity Copula). The function $M(u,v):[0,1]^2 \to [0,1]$ defined by

$$M(u,v) = \min(u \cdot v), \quad u,v \in [0,1],$$

is a copula called the copula of complete comonotonicity (see, picture 1b) (also called the upper Fréchet–Hoeffding bound).

c) (Countermonotonicity Copula**).** The function $W(u,v):[0,1]^2 \to [0,1]$ defined by

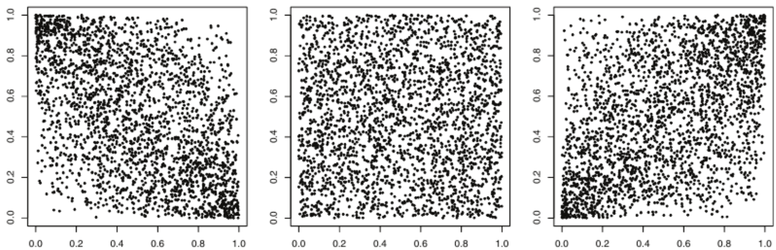
$$W(u,v) = \max(u+v-1,0), \quad u,v \in [0,1],$$

is a bivariate copula called the copula of complete countermonotonicity (see, picture 1c) (also called the lower Fréchet–Hoeffding bound).

d)(Gaussian (Normal) copula). The normal copula takes the form

$$C_\theta(u,v) = \Psi_\theta\left(\Phi^{-1}(u), \Phi^{-1}(v)\right) = \frac{1}{2\pi(1-\theta^2)^{1/2}} \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \exp\left\{\frac{-(s^2-2\theta st+t^2)}{2(1-\theta^2)}\right\} dsdt,$$

where Φ is the d.f. of the standard normal distribution, and Ψ is the standard bivariate normal distribution with correlation parameter $\theta \in (-1,1),\ \theta \neq 0.$ There are numerous applications of Gaussian copulas, particularly in engineering and finance. Scatterplots based on different parameters are visualized (see, picture 2).



Picture 2– Visual graph of normal copula scatterplots based on different parameters.

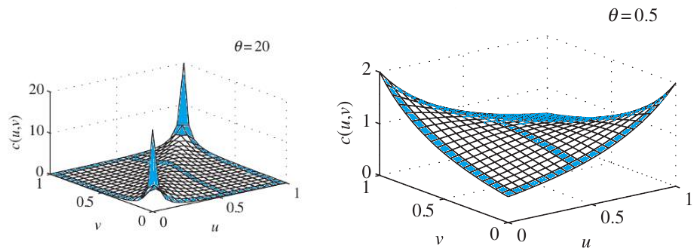
We can see from picture 2 scatterplot of 2500 samples from the bivariate Gaussian copula with $\theta = -0.5$ (left), 0 (middle), and 0.5 (right). Note that the dependence is increasing with θ ,and the case 0 corresponds to independence. Also note that the scatterplots have two axes of symmetry: (i) symmetry around the first diagonal (corresponding to exchangeability) and (ii) symmetry around the second diagonal (corresponding to the fact that the Gaussian copula is radially symmetric).

e) (Plackette Copula). The Plackette copula has been applied in recent years. The Plackette copula function is defined as follows

$$C_\theta(u,v) = \frac{[1+(\theta-1)(u+v)] - \sqrt{[1+(\theta-1)(u+v)]^2 - 4\theta(\theta-1)uv}}{2(\theta-1)}, \quad \theta > 0,\ \theta \neq 1.$$

If $\theta = 1$, then the Plackette copula function $C_\theta(u,v) = u \cdot v = \Pi(u,v)$. Taking the partial derivatives with respect to u and v, its copula density function can be written as follows

$$c_\theta(u,v) = \frac{\partial^2 C_\theta(u,v)}{\partial u \partial v} = \frac{\theta[1+(\theta-1)(u+v-2uv)]}{\left\{[1+(\theta-1)(u+v)]^2 - 4\theta(\theta-1)uv\right\}^{3/2}}.$$



Picture 3 – Visual plot of the Plackett copula density function when the parameter is 20 and 0.5.

From the copula density function plots with different parameters in picture 3, it is seen that (i) the density is higher if both u and v take on smaller or bigger values at the same time for $\theta = 20$, i.e., high follows high and low follows low as the representation of positive dependence; and (ii) the negative dependence is observed from the density function plot for $\theta = 0,5$, in this case, smaller u and bigger v reach higher density and conversely.

Copulas are powerful mathematical tools used in various fields, including finance, economics, and risk management, to model dependencies between random variables. They offer a flexible framework for capturing complex relationships that may not be adequately described by traditional correlation measures. In the context of economic systems and business processes, copulas can be applied in several ways (see, [2]-[5]):

– Risk Management: Copulas can be used to model dependencies between different types of risks, such as market risk, credit risk, and operational risk. By capturing the joint distribution of these risks, businesses can better understand their overall risk exposure and make more informed decisions regarding risk mitigation strategies.

– Portfolio Management: In finance, copulas are often employed to model the joint distribution of asset returns in a portfolio. By incorporating dependencies between asset returns, portfolio managers can more accurately assess portfolio risk and optimize asset allocation to achieve desired risk-return profiles.

– Insurance and Actuarial Science: Copulas are widely used in insurance and actuarial science to model dependencies between insurance claims and other relevant factors, such as policyholder characteristics and macroeconomic variables. This enables insurers to better assess their liabilities and set appropriate premium levels.

– Supply Chain Management: Copulas can be applied to model dependencies between different stages of the supply chain, such as demand forecasting, production scheduling, and inventory management. By understanding the joint distribution of these variables, businesses can optimize their supply chain operations to minimize costs and improve efficiency.

– Credit Risk Modeling: In banking and finance, copulas are used to model dependencies between default events of different counterparties in a credit portfolio. This is particularly important for assessing the overall credit risk of a portfolio and determining capital adequacy requirements under regulatory frameworks such as Basel III.

– Extreme Value Analysis: Copulas are useful for modeling extreme events, such as financial crises or large-scale disruptions in supply chains. By capturing the tail dependencies between variables, businesses can better understand the likelihood and potential impact of extreme events on their operations.

Overall, copulas provide a flexible and versatile framework for modeling dependencies in economic systems and business processes, allowing for more accurate risk assessment, portfolio management, and decision-making.

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