LIMITATIONS IN APPLICATION OF BILATERAL ESTIMATORS OF BASIC FREQUENCY IN ANALYSIS OF NATURAL VIBRATIONS OF CIRCULAR PLATES WITH VARIABLE THICKNESS AND CLAMBED EDGES

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Summary

In this paper the bilateral estimators of power variable thickness along the radius in diaphragm form where analyzed in a theoretical approach. The approximate solution of boundary problem of transversal vibration of plates with variable rigidity has been applied for chosen values of power indicator of variable thickness and material Poisson's ratio. The combinations of varying thickness and Poisson’s ratio has been chosen which led to exact form solutions [2, 3] Particular attention has been given to a singularity arising from the uncertainty of estimates of Berstein-Kieropian. The method developed in this paper allows has allowed obtaining the influence Cauchy function for arbitrary values of m and ν, which are physically justified. Therefore, the aim of a paper was to explore why for a plate above a certain value exact solution do not exist and FEM solution gives large discrepancies.

1. Introduction

In a previous paper [15] authors analyzed the use of simplest lower estimator to calculate the basic frequency of axi-symmetrical vibration of plates with variable thickness circular diaphragm type. The existence of the simplest estimator of the actual value of the parameter depending on the frequency rate of change characterized by thick plate (m = 3.25 to 5,999) was analyzed. The accuracy of the method differed from the FEM and in order to improve the accuracy of the estimators it was decided to use a higher order, in this case double. Using the bilateral estimator the similar problem arose in the calculation of exact solution in a paper by Conway [3]. It seems that the problem lies in the fact that diaphragms for meters which is close to 4 in the center of symmetry have a very low rigidity and on a boundary value it creates a hole in the middle with a radius of 1mm to 23mm. As a common known plate with a hole required 2 additional boundary conditions on the edge of the hole, the hole is a singularity which requires detailed analysis. It is widely known that clarification of the model leads to the complexity of solutions. The compromise between the possibilities of addressing coastal vibration and stability of mechanical systems and simplifying gives the total allowable use of methods of the influence functions, and partial discretization characteristic series. Good results achieved in previous publications [15], which included linear modeling of mechanical systems with discrete-continuous parameters encouraged authors to use the above-mentioned methods for studying vibration plates diaphragm type of ring. In this case, the influence of the function, which is the product of the Cauchy and Heaviside unit functions were applied. This function features the influence of derivatives, which are fundamental solutions of linear differential equations and can build their base of the integrated general solutions with various types of δ ratios in the Dirac. For the vibration test plate with varying parameters method of partial discretization has been used. It is based on the method of influence and has been previously proposed by L. Zoryj and J. Jaroszewicz to analyze vibration plates fixed and variable thickness with an additional mass focused [13]. The record for continuous or continuous-discrete mass distribution systems discrete can be replaced with one, two and n-degrees of freedom, which are characterized by the same function of stiffness. The plate’s mass focuses on the rings with a certain radius.

Total weight of the replacement system is equal to its own weight plates. This procedure uses a universal characteristic of the equation.

2. Formulation of the problem

R-radius circular plate having a clamped edge which cross section presented on Fig.1 has been considered.

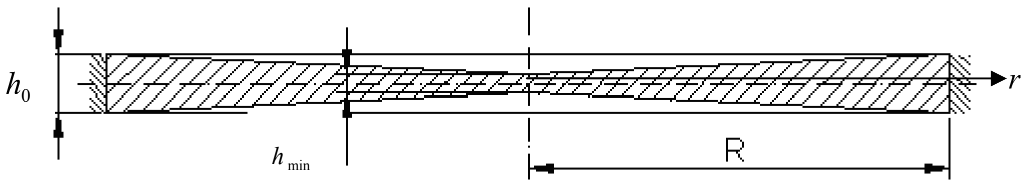
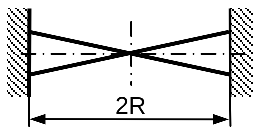
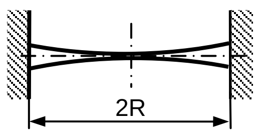


Figure 1: Cross section of the plate of diaphragm type

Table 1: Variable thickness plates

№	Diaphragm	Thickness	Schema
1.	Linear variable thickness	$h = h_0\left(\frac{r}{R}\right)$	
2.	Power variable thickness	$h = h_0\left(\frac{r}{R}\right)^{\frac{m}{3}}$ $2 < m < 6$	

Its thickness *h* and flexural rigidity *D* change in the following way (table 1):

$$h = h_0\left(\frac{r}{R}\right)^{\frac{m}{3}} \; ; \; D = D_0\left(\frac{r}{R}\right)^{m} \; ; \; 0 < r \leq R \, ; D_0 = \frac{Eh_0^3}{12\left(1-\nu^2\right)} \tag{2.1}$$

where $D_0, h_0, m \geq 0$ - are the constants, *r*- denotes the radial coordinate, *E* - a Young's modulus, and ν - denotes Poisson's ratio.

Investigation of free, axi-symmetrical vibrations of such a plate is reduced to an analysis of the boundary problem [2, 5, 11]

$$L_0[u] - pr^{-\frac{2}{3}m}u = 0 \tag{2.2}$$

$$u(R) = 0 \qquad u`(R) = 0 \tag{2.3}$$

where

$$L_0[u] \equiv u^{IV} + \frac{2}{r}(m+1)u^{III} + \frac{1}{r^2}(m^2 + m + \nu m - 1)u^{II} + \frac{1}{r^3}(m-1)(\nu m - 1)u^{I} \tag{2.4}$$

$$p = \frac{\rho h_0}{D_0} R^{\frac{2}{3}m} \varpi^2$$

$u = u(r)$ denotes the amplitude of flexural vibrations, ρ - density of the material of the plate, ω - the parameter of frequency /angular velocity/. Boundary value conditions corresponding to a clamped edge (2.3) has been defined as zero values of deflection and zero values of the angle of deflection for r=R. Additional conditions pertaining to the centre of symmetry of a plate (r=0) have limited values of deflection u(0) <∞ and zero values of the angle of deflection u'(0)=0. The value $m = 0$ refers to a plate with constant thickness; $m > 0$ - to plates of the diaphragm type with thickness decreasing toward the axial center ; $m < 0$ – to disc type plates with thickness increasing toward the axial center [5, 11]. The border of variation of the power index $m \geq 0$ has been determined, for which the most simple estimators of the basic frequency ω_1 exist and therefore can be calculated. i.e. it has been search for the lowest proper value of the border problem (2.2) ÷ (2.3). In the problem (2.2) ÷ (2.3), a limitation of solutions for r going to 0 and their first up to the third derivatives, with respect to the independent variable *r* is required [2].

General form of Cauchy function was proposed by L.M. Zoryj and J. Jaroszewicz in older works.

$$K_0(r,\alpha) = \frac{1}{(1-\nu)m}\left[\frac{1}{2\sqrt{D}}\left(r^{S_3}\alpha^{S_n+m+1} - r^{S_4}\alpha^{S_3+m+1}\right) - \frac{1}{2-m}\left(r^{2-m}\alpha^{m+1} - \alpha^3\right)\right] \tag{2.5}$$

where: s_1, s_2, s_3, s_4 - routs of characteristic equation, *D* – determinant of square equation

$$s^2 - (2-m)s - m(1-\nu) = 0 \tag{2.6}$$

$$D = \left(1 - \frac{m}{2}\right)^2 + m(1-\nu) \tag{2.7}$$

Equation (2.6) received from characteristic equation

$$s\{s^3 + 2(m-2)s^2 + \left[4 - 5m + m^2 + m\nu\right]s + m(2-m)(1-\nu)\} = 0, \tag{2.8}$$

which roots present in next form:

$$S_1 = 0,\ S_2 = 2-m,\ S_{2,4} = -\frac{m}{2} + 1 \pm \sqrt{D} \tag{2.9}$$

So, for $L_o[u] = 0$ the fundamental system of solutions, in those cases is (according to Euler's theory of equations) as follows:

$m = 0$

$1,\quad \ln r,\quad r^2,\quad r^2 \ln r;$

m = 2 (2.10)

$1,\quad \ln r,\quad r^{\sqrt{2-2\nu}},\quad r^{-\sqrt{2-2\nu}}$

Based on those calculations, the following remarks can be formulated:

1. All roots of the cubic equation (2.8), as well as those of the quadratic equation (2.6) are real numbers for any given physically justified values of *m* - power indexes and values of Poisson's ratio ν ($\nu \in (0;0{,}5)$).

2. The equation (2.6) has no other multiple (repetitive) roots for $m \in (-\infty, +\infty)$ and $\nu \in (0;0{,}5)$.

3. The fundamental systems of solutions for Euler's differential equations $L_o[u] = 0$ posses logarithmic peculiarities only in cases m = 0, m = 2 and are determined by the formulas (2.10); in all of the rest cases they have power character peculiarities.

3. Derivation of the frequency equation in particular case $\nu = \frac{1}{m}$

The necessary limited solution of equation (2.2) will be derived according to the known formula [11, 15],

$$S_j = s_{j0} + ps_{j1} + p^2 s_{j2} + ..., \quad (j = 1,2) \tag{3.1}$$

where

$$S_{jk} = \int_0^r K(r,\tau)\tau^{-\frac{2}{3}m} s_{j,k-1}(\tau)d\tau \tag{3.2}$$

k = 1, 2, ...; S_{10}, S_{20} – the proper solutions (limited for r = 0) of Euler`s equation $L_0[u] = 0$. Therefore K(r, τ) – its Cauchy function, is the solution of the equation $L_0[u] = 0$, which satisfies the conditions:

$$K(r,\tau) = K'(r,\tau) = K''(r,\tau) = 0 \qquad K'''(r,\tau) = 1 \tag{3.3}$$

In order to build the above mentioned function K(r, τ), the solutions to the given equation, which correspond to the operator (2.4) is necessary. Substituting $u = r^s$ for p = 0 in (2.2), an appropriate algebraic equation in respect to the parameter s [5] has been obtained:

$$s\{s^3 + 2(m-2)s^2 + [4 - 5m + m(m+\nu)]s + m(2-m)(1-\nu)\} = 0 \tag{3.4}$$

The roots of this equation and the corresponding Cauchy function were previously determined for a few values of m where m ≤ 3 [2]. For that reason in a current paper the following case has been examined:

$$2 \le m < \infty \qquad \nu = \frac{1}{m} \tag{3.5}$$

The parameter m is this same which is used in the operator (2.4). In the case of a constant thickness plate m = 0, the coefficient of natural frequency γ(m), described by the formula (4.3), is not dependent on ν, what was showed in the following papers: [2,5,11].

Obviously, for the operator (2.4) the coefficient of u' equals zero and the roots of the equation (3.4) are as follows:

$$s_1 = 0 \qquad s_2 = 1 \qquad s_3 = 1 - m \qquad s_4 = 2 - m \tag{3.6}$$

where s_3 and s_4 are determined from the square equation, which was obtained from (3.4) for $s_1 = 0$ and $s_2 = 1$).

Hence, the corresponding fundamental system (3.6) of solutions of equation $L_0[u] = 0$ takes the following form:

$$u_1 = 1 \qquad u_2 = r \qquad u_3 = r^{1-m} \qquad u_4 = r^{2-m} \tag{3.7}$$

An application of the above system leads to the formula:

$$K(r,\tau) = (1-m)^{-1}\left[m^{-1}\left(-r\tau^2 + r^{1-m}\tau^{2+m}\right) + (2-m)^{-1}\left(r^{2-m}\tau^{1+m} - \tau^3\right)\right] \tag{3.8}$$

Now the solutions can be constructed (3.1). After calculating the first two integrals of (3.2) (for j = 1, 2 and k = 1) the following relations have been obtained:

$$S_{10} = 1 \qquad S_{20} = r$$

$$S_1(r) = 1 + A_1 pr^a + A_2 p^2 r^{(2a)} \qquad S_2(r) = r + B_1 pr^{a+1} + B_2 p^2 r^{(1+2a)} \tag{3.9}$$

$$A_1 = \left[ab(b+1)(a-1)\right]^{-1} \qquad B_1 = \left[a(a+1)(b+1)(b+2)\right]^{-1} \tag{3.10}$$

$$A_2 = \frac{1}{2}A_1 \cdot \left[a(2a-1)c(c+1)\right]^{-1} \qquad B_2 = \frac{1}{2}B_1 \cdot \left[a(2a+1)(c+1)(c+2)\right]^{-1} \tag{3.11}$$

where

$$a = 4 - \frac{2}{3}m \quad \text{and} \quad b = 2 + \frac{m}{3} \quad \text{and} \quad c = 6 - \frac{m}{3}$$

After taking into consideration the formulas for u_3 and u_4 for the conditions (3.9), a conclusion can be drawn that the limited solutions of equation (2.2) are determined by the formula:

$$u(r) = C_1 S_1(r) + C_2 S_2(r) \tag{3.12}$$

where C_1 and C_2 – are arbitrary constants. After inserting this solution into the condition (3.5), the homogenous system of two algebraic equations for the constants C_1 and C_2, has been obtained where its determinant is the left part of the frequency problem (2.2) – (2.3). Therefore:

$$\begin{vmatrix} S_1(r) & S_2(r) \\ S_1'(r) & S_2'(r) \end{vmatrix} = 0 \tag{3.13}$$

Considering the formulas (3.9) – (3.11) the first approximation (accurate to p, hence to the square of frequency) has been gained, which takes the following form:

$$1 - a_1 pR^a + a_2 p^2 R^{2a} - \ldots = 0 \tag{3.14}$$

where

$$a_1 = -\left[A_1(1-a) + B_1(1+a)\right] \tag{3.15}$$

$$a_2 = \left[A_1 B_1 + A_2(1-2a) + B_2(1+2a)\right] \tag{3.16}$$

Taking into consideration the equation:

$$pR^a = \frac{\rho h_0}{D_0} R^4 \omega^2 \tag{3.17}$$

Neglecting second terms of series a_2 it can be transformed (3.16) to the following form:

$$1 - a_1 \frac{\rho h_0}{D_0} R^4 \omega^2 = 0 \tag{3.18}$$

Remarks:

For m = 3, ν = ⅓ we have obtained $A_1 = (24)^{-1}$, $A_2 = (24 \cdot 360)^{-1}$, $B_1 = (120)^{-1}$, $B_2 = (120 \cdot 840)^{-1}$ [11].

Formulas (3.10), (3.11) are in accordance with the values quoted in the above remarks.

Obviously first of solution (3.9) and second derivative of solutions (3.9) for $a = 0 \quad (m = 6)$ are independent of r, in this case coefficients (3.10) – (3.11) lose sense for $m \to 6$, $B_1 \to \infty$, $B_2 \to \infty$. Besides $A_1 \to \infty$ for $m \to 4{,}5 \quad (a \to 1)$ and $A_2 \to \infty$ for $m \to 5{,}25 \quad (a \to 0{,}5)$

For cases (3.5) using (3.10) and (3.11) we transform (3.17) to the following form:

$$a_1 = 2[ab(b+1)(b+2)]^{-1} \tag{3.19}$$

4. The bilateral estimators for basic frequency in particular case $\nu = \dfrac{1}{m}$

Take into account the series (3.15) known Bernstein estimators with the following form [1] can be applied.

$$\left(a_1^2 - 2a_2\right)^{-\frac{1}{4}} < \gamma < \sqrt{2} \cdot \left(a_1 + \sqrt{a_1^2 - 4a_2}\right)^{-\frac{1}{2}} \tag{4.1}$$

Coefficients of the series a_1, a_2 dependent on m can be constructed on the basis of (3.15). Coefficient a_1 scrutinized in previous work, where exact formula has been constructed.

$$a_1 = \frac{9^4 \cdot 3 \cdot 10}{[(6-m)\cdot(6+m)\cdot(9+m)\cdot(12+m)]} \tag{4.2}$$

To develop formula (3.16) for a_2 in a similar form formula in following form should be present.

$$A_1 B_1 + A_2(1-2a) \equiv A_1 \cdot \frac{1}{a} \cdot F(m) \tag{4.3}$$

where

$$F(m) = [(a+1)\cdot(b+1)\cdot(b+2)]^{-1} - \frac{1}{2}[c\cdot(c+1)]^{-1} \tag{4.4}$$

Second addition in expression (3.16) can be present in form

$$B_2(1+2a) = \frac{1}{2} B_1 \cdot f(m) \tag{4.5}$$

where

$$f(m) = [a\cdot(c+1)\cdot(c+2)]^{-1} \tag{4.6}$$

Now exploit determination (5.4)-(5.6) also formulas (3.10)-(3.11) we constructed coefficient a_2 record in form expression

$$a_2 = \frac{1}{a^2\cdot(b+1)}\left\{\frac{F(m)}{(a-1)\cdot b} + \frac{1}{2\cdot(a+1)\cdot(b+2)\cdot(c+1)\cdot(c+2)}\right\} \tag{4.7}$$

Take into account identity

$$m - 4{,}5 = \frac{3}{2}(1-a)$$

The first component sum (5.7) in form has been found

$$\frac{F(m)}{(a-1)\cdot b} = -\frac{81\cdot(m-3)\cdot(m+24)}{2\cdot(15-2m)\cdot(9+m)\cdot(6+m)\cdot(12+m)\cdot(18-m)\cdot(21-m)} \tag{4.8}$$

Considering also identity

$$\frac{1}{2\cdot(a+1)\cdot(b+2)\cdot(c+1)\cdot(c+2)} = \frac{81}{2\cdot(12+m)\cdot(15-2m)\cdot(21-m)\cdot(24-m)} \tag{4.9}$$

Finally from (5.7) general form second coefficients of characteristic series a_2 has been recived

$$a_2 = \frac{3^9\cdot 5}{2\cdot(12+m)\cdot(6+m)\cdot(18-m)\cdot(21-m)\cdot(24-m)\cdot(6-m)^2\cdot(9+m)^2} \tag{4.10}$$

Example results of calculation for changed cases $2 < m < 6$ and $\frac{1}{\nu} = m$ obtain on the basis of

formulas (5.1), (5.2) and (5.10) present on the table 2,3. and on fig. 1.

Where mean arithmetic value of basic frequency coefficient for m = 3, ν = ⅓ is:

$$\gamma = \frac{1}{2}(\gamma_- + \gamma_+) = \frac{1}{2}(8{,}6505 + 8{,}8429) = 8{,}7467 \tag{4.11}$$

the basic frequency can be found

$$\omega = \gamma\cdot\frac{1}{R^2}\sqrt{\frac{D_0}{\rho h_0}} \equiv \gamma\frac{h_0}{R^2}\sqrt{E\left[12\rho\left(1-\nu^2\right)\right]^{-1}} \tag{4.12}$$

5. Discuses of results of calculations

Results of calculations where compared with previous paper [15] in table 2 and fig 2 which were calculated by means Cauchy function and characteristic series method using simplest estimator and exact solution recived on base Bessel special function by Conway, Hondkiewic, Kovalenko.

Table 2: Results of calculation of base frequency

No.		1.	2.	3.	4.	5.	6.	7.	8.	9.
Coefficient m		2	2,5	3	3,7	3,9755	4,4999	5	5,5	5,9999
Two first terms of characteristic series	a_1	1/61	1/61	1/60	1/55	1/52	1/43	3/97	1/18	250
	a_2	2/58941	3/75050	1/20160	6/79567	1/10684	8/50175	19/55929	13/10017	31250,21
Simplest lower estimator	$\gamma_p(m) = (a_1)^{-\frac{1}{2}}$	7,80	7,83	7,75	7,41	7,19	6,58	5,69	4,24	0,06
The double sides estimator	γ_-	8,38	8,56	8,65	8,62	8,55	8,27	7,75	6,73	0,83
	γ_+	8,45	8,66	8,84	9,19	10,16	-	-	-	-
	γ_0	8,42	8,61	8,75	8,91	9,36	-	-	-	-
Value of exact solution	γ_{ext}	8,46	8,60	8,75	-	-	-	-	-	-

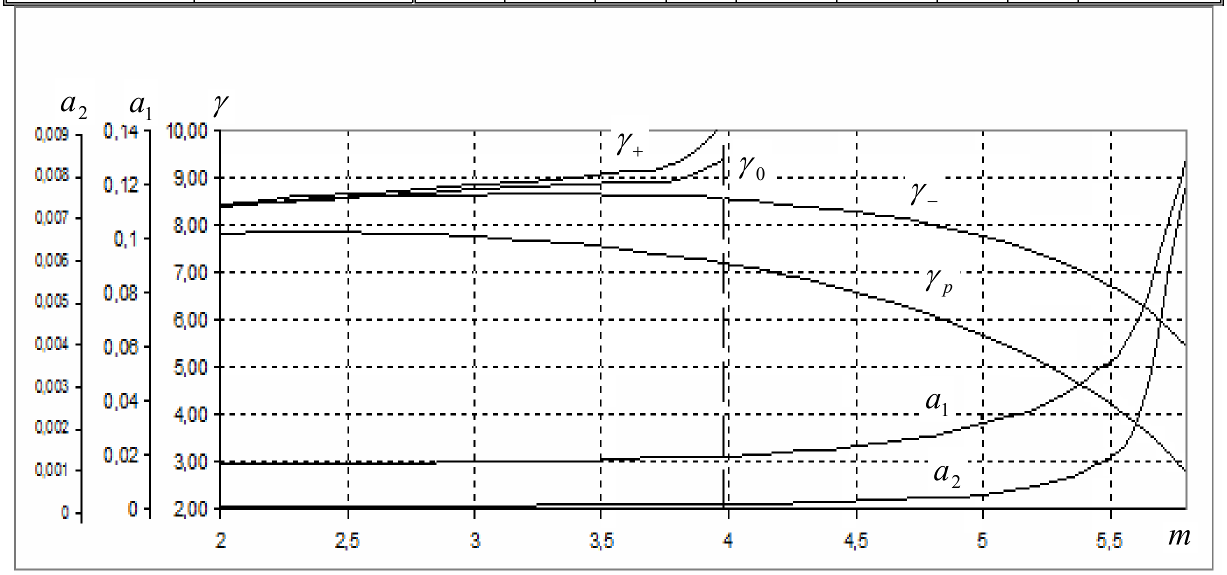


Figure 2: The curve showing influence of the plate thickness index on the bilateral estimators of the frequency coefficients

Boundary values, for which the upper estimator does not exist can be settled on the basis of the investigation of under roots expression form change (5.1) $(a_1^2 - 4a_2)$ with $m = 3{,}97$ change sign form positive to negative with $m = 3{,}98$ so calculating accurate values a_1 and a_2 on the base of base (5.10) we have respectively:

$$(a_1^2 - 4a_2)\Big|_{m=3,97} = 7 \cdot 10^{-7} > 0$$

$$(a_1^2 - 4a_2)\Big|_{m=3,98} = -6 \cdot 10^{-7} < 0$$

Under estimated values there for it can not be even used in approximate application however the simplest lower estimator can be applied to preliminary engineering calculations for constant and for variable thickness plates when $0 \le m < 4$.

Table 3: The base frequency of the plate with nonlinearly variable thickness $h(r \to 0) = 0{,}01mm$ [15]

№	Material	ν	Coefficient m	D_0 [Nm]	Masses [kg]	Average of bilateral estimator γ_0	Frequency of the bilateral estimator f[Hz]	Frequency in the FEM analysis f[Hz]	Difference Δ %
1.	titanium	0,36	2,78	9778	6,093	8,697682	322,09 Hz	330,19 Hz	2,45
2.	steel	0,27	3,7	18560	9,834	8,90738	340,23 Hz	347,15 Hz	1,95
3.	zinc	0,25	4,0	10531	8,410	9,356813	286,75 Hz	267,35 Hz	-7,26
4.	concrete	0,17	5,9	1716	1,983	unknown	unknown	211,76 Hz	unknown

It should be noticed that in the case of a constant thickness plate $(m = 0)$, the multiplier $\gamma = \gamma_0 = (3{,}196)^2 = 10{,}2122$ is independent from ν [8, 4]. So the ratio of coefficients:

$$\frac{\gamma_0}{\gamma} = \frac{10{,}2144}{8{,}7467} = 1{,}1678 \approx 1{,}17\,, \tag{4.13}$$

It does agree with results of calculations obtained by Conway in [2] and presented in introduction to this paper ($T/T_0 = 1{,}17$). Continuing, in analogical fashion the basic frequency for other combinations of m and ν has been calculated. The results of calculations are presented below in table 4

In table 4 are presented values revived by Jaroszewicz J. Zoryj L. cases for witch Conway derive characteristic function on base special Bessel function. Ratio $\frac{\gamma_0}{\gamma}$ of natural frequency plate co constants thickness and plate of variable thickness for Conway values m , ν . This table contain model value of solution with we compare approximate results.

Table 4: The results of calculations [11]

L.p.	m	ν	$\frac{\gamma_0}{\gamma}$	$\frac{T}{T_0}$
1	2	$\frac{1}{9}$	$\frac{10{,}2144}{9{,}4562} = 1{,}0802$	1,0824
2	$\frac{18}{7}$	$\frac{5}{21}$	$\frac{10{,}2144}{9{,}0777} = 1{,}1252$	1,1261
3	$\frac{18}{7}$	$\frac{7}{18}$	$\frac{10{,}2144}{8{,}6376} = 1{,}1825$	Conway does not applying exact solution [2]

In table 4 Conwey could not apply the exact method [2], because the condition: $\nu = \frac{2m-3}{9}$ was not fulfilled.

6. Conclusions

On the basis of Figure 3 it can be seen that the simplest estimator Yp gets underestimates results the value with the increase of the coefficient m, the error increases in relation to the value of science, and FEM results obtained in previous work. In contrast to double estimate its give accurate results, this follows from the analysis of the value of the denominator of the estimator Y +, the value calculation is consistent with the exact solution for m = 3, is 8.75 the same is true for m = 2 (8.40) im = 2 , 5 (8.60). Deriving of the above mentioned formulas for the Cauchy functions, as well as fundamental systems of function operator $L_o[u]$ allows to study of the convergence problem (velocity of convergence) of solutions of equation (2.4) in form of power series in respect to parameters of frequencies, depending on values of parameters *m* and ν.

Having the influence functions of operator $L_o[u]$ corresponding solutions and use them for any given physically justified values of parameters *m* and ν ($m \in (-\infty, +\infty)$; $\nu \in (0; 0{,}5)$) can be consequently determined, when the exact solutions are unknown on base general form of Cauchy function (2.5). On the basis of quoted solutions, simple engineering formulas for frequencies estimators of circular plates, which are characterized by variable parameters distribution, can be derived and limits of their application can be identified. The bilateral estimators calculated using four first elements of the series, allows to credibly observe an influence material's constants: Young modulus- E, Poisson ratio- ν, density - ρ on the frequencies on axi-symmetrical vibrations of circular plates, which thickness or rigidity changes along the radius according to the power function.

The bilateral estimator underrate values, application of bilateral estimator significantly improves the accuracy of calculations terms of exacts solutions.

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