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Damping of a Solution of Linear Autonomous Difference–Differential Systems with Many Delays Using Feedback

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Abstract—For linear autonomous difference–differential systems with commensurable delays, the problem of damping the solution by using a linear difference–differential controller with a state feedback is solved. A generalization of these results to linear autonomous difference–differential systems of neutral type with commensurable delays in the case of a continuous solution is proposed. A distinctive feature of the present work is that the initial system is not completely controllable.

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INTRODUCTION

A key issue of automatic control theory is constructing controllers that ensure prescribed properties for a system. In this connection it is necessary to consider the problems of stabilization [1–4], modal controllability [5, 6], spectral reducibility [7, 8], and complete controllability using a feedback [9–11]. Consider the last-named problem in greater detail.

The problem of complete controllability (complete damping) had been originally stated by Krasovskii [12] for systems with time delay and then was considered by many researchers (historical data can be found in [13, 14], and they are not discussed in the present paper). In [13–15], a generalization of this problem in the sense of damping the system's solution by using a constantly control was proposed. For the most part, the results of investigating the complete controllability problem and its generalizations [13–15] are solvability criteria and methods for constructing open-loop controls. As an exception we point out to [9–11], where a one-input linear difference–differential system is damped by a feedback. The basic idea is to ensure the pointwise singularity of a closed-loop system in the directions corresponding to phase variables of the initial system by using a a complete damping controller. In this case, the necessary and sufficient condition of the existence of such a controller is the complete controllability condition [16], which coincides with the spectral controllability condition [17] (the complete controllability of a finite-dimensional subsystem that corresponds to every spectral value of the initial system).

In the case of multi-input systems with many delays in control the complete (spectral) controllability conditions are not needed for the existence of an open-loop control that damps the solution [13–15]. Consequently the question arises of whether it is possible to close a not completely controllable system by a linear feedback in such a way as to ensure for the initial system the equality $x(t) \equiv 0$, $t \geq t_1$ no matter what the initial state of the system may be. In the present paper, we obtain conditions for parameters of a system with commensurable delays under which the answer to this question is positive. The sufficient existence conditions of such a feedback coincide with the criterion of damping of incompletely controllable systems solution [14, 15].

The paper is organized as follows. In Sections 1–3, for a multi-input linear difference–differential system, we construct a linear controller that ensures that the solution is damped in the case where the complete controllability condition is violated. In Section 4, the proposed procedure is generalized to the case of a system of neutral type with a continuous solution.

1. STRUCTURE OF CONTROLLER FOR SYSTEMS WITH TIME DELAYS

Suppose that a plant is described by the linear autonomous difference–differential system with commensurable delays

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - ih) + \sum_{i=0}^{m} B_i u(t - ih), \quad t > 0 \tag{1.1}$$

and the initial condition

$$x(t) = \eta(t), \quad u(t) \equiv u^0(t), \; t \in [-mh, \; 0], \tag{1.2}$$

where $x \in \mathbb{R}^n$ is a solution of Eq. (1.1), $u \in \mathbb{R}^r$ is a piecewise continuous control, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times r}$, $i = \overline{0,m}$, are constant matrices, and $h > 0$ is a constant delay. Assume that in initial condition (1.2) the function $\eta \in C([-mh,0],\mathbb{R}^n)$, where $C([-mh,0],\mathbb{R}^n)$ is the space of continuous vector functions on the interval $[-mh,0]$ with values in $\mathbb{R}^n$, and $u^0(t)$, $t \in [-mh,0]$ is an arbitrary piecewise continuous function.

If we define the polynomial matrices

$$A(z) = \sum_{i=0}^{m} A_i z^i, \quad B(z) = \sum_{i=0}^{m} B_i z^i$$

and assume that z is the shift operator (i.e., $zf(t) = f(t-h)$), then system (1.1) can be rewritten in the operator form as $\dot{x}(t) = A(z)x(t) + B(z)u(t)$. For convenience, use this notation below.

The complete controllability (complete damping) criterion for system (1.1) has the form [16]

$$\operatorname{rank}[\lambda E_n - A(e^{-\lambda h}), B(e^{-\lambda h})] = n \;\forall\; \lambda \in \mathbb{C}, \tag{1.3}$$

where $E_k \in \mathbb{R}^{k\times k}$ is the identity matrix and $\mathbb{C}$ is the set of complex numbers. Recall that system (1.1) is called completely controllable if, for every initial condition (1.2), there exists a time $t_1 > 0$ and a control $u(t)$, $t \in (0, \; t_1 - mh]$, $u(t) \equiv 0$, $t > t_1 - mh$ such that

$$x(t) \equiv 0, \quad t \geq t_1. \tag{1.4}$$

As is shown in [9–11], for systems (1.1) with a scalar control without delays under the condition of complete controllability (1.3), identity (1.4) can be ensured by a state feedback controller. However, the controllers presented in [9–11] are not extended automatically to the general case of systems (1.1). There are two reasons for this. First, for systems of the general form (1.1), criterion (1.3) is not a necessary condition for the existence of an open-loop control without the requirement $u(t) \equiv 0$, $t > t_1 - mh$, which ensures (1.4) [14, 15]. Second, if condition (1.3) is violated, the input feedback control must change its structure according to a certain difference equation; below this issue is considered in more detail (see Theorem 1).

The problem of choosing a control $u(t)$, $t \geq 0$ that ensures identity (1.4) without the requirement $u(t) \equiv 0$, $t > t_1 - mh$ will be called, in contrast to the complete damping problem [9–12, 14, 16], the problem of damping the system solution.

In the present work, it is proposed to damp a system solution by a linear difference–differential controller of the form

$$u(t) = K_1(z)x(t) + e_1 x_{n+1}(t) + T\psi(t), \quad t > 0, \tag{1.5}$$

$$\psi(t) = Sz\psi(t) + K_2(z)x(t), \quad t > 0, \tag{1.6}$$

$$\dot{x}_{n+1}(t) = F_1^1(z)x(t) + F_2^1(z)x_{n+1}(t) + F_3^1(z)y(t), \; t > 0, \tag{1.7}$$

$$\dot{y}(t) = F_1^2(z)x(t) + F_2^2(z)x_{n+1}(t) + F_3^2(z)y(t), \; t > 0, \tag{1.8}$$

where $K_1(z) \in \mathbb{R}^{r\times n}[z]$ and $K_2(z) \in \mathbb{R}^{r_T\times n}[z]$ ($\mathbb{R}^{k_1\times k_2}[z]$ is the set of $k_1 \times k_2$ matrices whose elements are polynomials of the variable z), $T \in \mathbb{R}^{r\times r_T}$, $S \in \mathbb{R}^{r_T\times r_T}$, r_T is a certain natural number, $e_1 = \operatorname{col}[1,0,\ldots,0] \in \mathbb{R}^r$, $F_1^1(z) \in \mathbb{R}^{1\times n}[z]$, $F_2^1(z) \in \mathbb{R}^{1\times 1}[z]$, $F_3^1(z) \in \mathbb{R}^{1\times s}[z]$, $F_1^2(z) \in \mathbb{R}^{s\times n}[z]$, $F_2^2(z) \in \mathbb{R}^{s\times 1}[z]$, $F_3^2(z) \in \mathbb{R}^{s\times s}[z]$; $x_{n+1} \in \mathbb{R}$, $\psi \in \mathbb{R}^{r_T}$, and $y = \operatorname{col}[y_1,\ldots,y_s] \in \mathbb{R}^s$ are additional variables. The functions $x(t)$, $t < -mh$, $x_{n+1}(t)$, $y(t)$, and $\psi(t)$ $(t \leq 0)$ can be arbitrary continuous functions.

R e m a r k 1. We describe more precisely how the control is formed by controller (1.5)–(1.8). On each fixed half-interval $(lh, \; (l+1)h]$ $(l = 0,1,\ldots)$, the control $u(t)$, $t > 0$ is linearly expressed in terms of x, x_{n+1}, and y using (1.5) and difference equation (1.6). We substitute this control into (1.1), supplement the resultant relation with Eqs. (1.7) and (1.8), and obtain on the half-interval $(lh,\ (l+1)h]$ $(l = 0, 1, \ldots)$ a linear autonomous differential system with the solution $\mathrm{col}[x, x_{n+1}, y]$.

We formulate the solvability conditions for the problem of damping the system's solution and describe the matrices T and S that appear in the structure of controller (1.5)–(1.8). For this purpose, by analogy with [13–15], we consider the sequence of vectors δ_k, $k = m, m+1, \ldots$ that is the solution of the difference equation

$$B_0\delta_k + \sum_{i=1}^{m} B_i\delta_{k-i} = 0, \quad k = m, m+1, \ldots \tag{1.9}$$

produced by the initial condition $\delta_i = \tilde{\delta}_i$, $i = \overline{0, m-1}$.. The sequence δ_k, $k = m, m+1, \ldots$ exists if and only if (see [13–15]) $\tilde{\delta}_{m-i} = T_i c$, $i = \overline{1, m}$, where $T_i \in \mathbb{R}^{r \times r_T}$ are certain matrices and $c \in \mathbb{R}^{r_T}$ is an arbitrary constant vector (the same for all matrices T_i). A procedure for constructing the matrices T_i is presented in [13–15], and it is not described here. Note that its implementation is always possible and consists in solving the finite number of homogeneous algebraic systems. In (1.5) and in the subsequent discussion, we assume that $T = T_m$.

We find an arbitrary matrix $S \in \mathbb{R}^{r_T \times r_T}$ that satisfies the equations

$$B_0T_1S + \sum_{i=1}^{k} B_iT_i = 0, \quad T_kS = T_{k-1}, \quad k = \overline{2, m}.$$

The existence of the matrix S follows from the definition of the matrices T_i. Note that

$$\sum_{i=0}^{m} B_iTS^{m-i} = 0. \tag{1.10}$$

Define matrices $G_0 = B_0T$ and $G_i = G_{i-1}S + B_iT$, $i = \overline{1, m}$. We take into account that

$$G_m = \sum_{i=0}^{m} B_iTS^{m-i} = 0.$$

Let us introduce the notation

$$G(z) = \sum_{i=0}^{m-1} G_iz^i.$$

The following result is true [14].

T h e o r e m 1 (the criterion of damping the system solution). In order for any initial condition (1.2) of system (1.1) to have a control $u(t)$, $t > 0$ that ensures (1.4), it is necessary and sufficient that the following equality holds:

$$\mathrm{rank}[\lambda E_n - A(e^{-\lambda h}), B(e^{-\lambda h}), G(e^{-\lambda h})] = n \ \forall\ \lambda \in \mathbb{C}. \tag{1.11}$$

In what follows we suppose that condition (1.11) holds. Now we proceed to constructing controller (1.5)–(1.8) that ensures identity (1.4).

2. THE CONTROLLER CONSTRUCTION PROCEDURE

First, we explain the influence of the function ψ, which appears in the structure of controller (1.5)–(1.8), on the dynamics of the closed-loop system. For this purpose, we prove the following lemma, in which we incidentally specify the exact form of a system satisfied by the functions x, x_{n+1}, y in the case of controller (1.5)–(1.8).

L e m m a. Assume that $K_i(z)$, $i = 1,2$ and $F_i^j(z)$, $i = \overline{1,3}$, $j = 1,2$ are arbitrary polynomial matrices of the size indicated above. For all $\psi(t)$, $x(t)$, $x_{n+1}(t)$, $y(t)$ $(t \le 0)$, the functions $x(t)$, $x_{n+1}(t)$, $y(t)$ for $t > mh$ satisfy the linear autonomous difference-differential system with commensurable delays

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_{n+1}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} A(z) + B(z)K_1(z) + G(z)K_2(z) & B(z)e_1 & 0_{n\times s} \\ F_1^1(z) & F_2^1(z) & F_3^1(z) \\ F_1^2(z) & F_2^2(z) & F_3^2(z) \end{bmatrix} \begin{bmatrix} x(t) \\ x_{n+1}(t) \\ y(t) \end{bmatrix}, \tag{2.1}$$

where $0_{n\times s} \in \mathbb{R}^{n\times s}$ is a zero matrix.

P r o o f. Suppose that $\psi(t)$, $t > 0$ is determined by Eq. (1.6). Using the definition of the matrices G_i we make the following transformations:

$$B(z)T\psi(t) = \sum_{i=0}^{m} B_i z^i T\psi(t) = G_0\psi(t) + \sum_{i=0}^{m-1} (G_{i+1} - G_i S) z^{i+1}\psi(t) = \sum_{i=0}^{m} G_i z^i \psi(t) - \sum_{i=0}^{m-1} G_i z^{i+1} S\psi(t)$$

$$= G(z)(E_{r_T} - zS)\psi(t).$$

In the last equality, we used the fact that $G_m = 0$. In view of (1.6), we finally come to the equality

$$B(z)T\psi(t) = G(z)K_2(z)x(t). \tag{2.2}$$

Now we substitute the control $u(t)$, $t > 0$ determined by formula (1.5) into system (1.1) and replace $B(z)T\psi(t)$ in accordance with (2.2). As a result we obtain that the solution $\operatorname{col}[x(t), x_{n+1}(t), y(t)]$ $(t > mh)$ of system (1.1) closed by controller (1.5)–(1.8) satisfies (2.1). The lemma is proved.

Let us describe the procedure for constructing controller (1.5)–(1.8) that ensures identity (1.4) for the solution of system (1.1). Let $\tilde{B}(z) = [B(z),\ G(z)]$, and consider a couple of matrices $\{A(z), \tilde{B}(z)\}$. In view of (1.1) we have

$$\operatorname{rank}[\tilde{B}(z), \ldots, A^{n-1}(z)\tilde{B}(z)] = n$$

(hereinafter, by the rank of a polynomial matrix we mean [18] the maximum order of its minor that is not identical zero). We choose columns $\tilde{b}_{s_i}(z)$, $i = \overline{1,\theta}$ of the matrix $\tilde{B}(z)$ in such a way that the following equalities hold:

$$\operatorname{rank}[\tilde{b}_{s_1}(z), \ldots, A^{n_1-1}(z)\tilde{b}_{s_1}(z), \tilde{b}_{s_2}(z), \ldots, A^{n_2-1}(z)\tilde{b}_{s_2}(z), \ldots, A^{n_j-1}(z)\tilde{b}_{s_j}(z)]$$

$$= \operatorname{rank}[\tilde{b}_{s_1}(z), \ldots, A^{n_1-1}(z)\tilde{b}_{s_1}(z), \tilde{b}_{s_2}(z), \ldots, A^{n_2-1}(z)\tilde{b}_{s_2}(z), \ldots, A^{n_j-1}(z)\tilde{b}_{s_j}(z), A^{n_j}(z)\tilde{b}_{s_j}(z)]$$

$$= n_1 + n_2 + \ldots + n_j,\ \ j = \overline{1,\theta},\ \ n_1 + n_2 + \ldots + n_\theta = n.$$

Define

$$A_{\tilde{b}}(z) = [\tilde{b}_{s_1}(z), \ldots, A^{n_1-1}(z)\tilde{b}_{s_1}(z), \tilde{b}_{s_2}(z), \ldots, A^{n_2-1}(z)\tilde{b}_{s_2}(z), \ldots, \tilde{b}_{s_\theta}(z), \ldots, A^{n_\theta-1}(z)\tilde{b}_{s_\theta}(z)].$$

Taking into consideration that rank $\operatorname{rank} A_{\tilde{b}}(z) = n$, we construct [18] a square polynomial matrix $R(z)$, $\det R(z) \equiv \text{const} \ne 0$ such that the matrix $R(z)A_{\tilde{b}}(z)$ has the structure

$$R(z)A_{\tilde{b}}(z) = \begin{bmatrix} 0 & 0 & \ldots & * \\ \ldots & \ldots & \ldots & \ldots \\ 0 & * & \ldots & * \\ * & * & \ldots & * \end{bmatrix},$$

where the symbol * denotes some polynomials; here, polynomials on the second diagonal are different from identical zero. Note that multiplying the matrix $A_{\tilde{b}}(z)$ by the matrix $R(z)$ on the left is equivalent to elementary transformations of its rows.

Let $\bar{A}(z) = R(z)A(z)R^{-1}(z)$ and $\bar{B}(z) = R(z)\tilde{B}(z)$. For simplicity, assume that $s_1 = 1$ in the matrix $A_{\tilde{b}}(z)$. Then, the first column $\bar{b}_1(z)$ of $\bar{B}(z)$ has the form $\operatorname{col}[0, \ldots, 0, \bar{b}(z)]$, where $\bar{b}(z)$ is a polynomial.

Since equality (1.11) holds, $\operatorname{rank}[\lambda E_n - \bar{A}(e^{-\lambda h}), \bar{B}(e^{-\lambda h})] = n\ \forall\ \lambda \in \mathbb{C}$. In view of the last condition, we can construct (see [19]) a polynomial matrix $\bar{K}(z)$ such that

$$\operatorname{rank}[\lambda E_n - D(e^{-\lambda h}), \bar{b}_1(e^{-\lambda h})] = n\ \forall\ \lambda \in \mathbb{C}, \tag{2.3}$$

where $D(z) = \bar{A}(z) + \bar{B}(z)\bar{K}(z)$.

Consideration the auxiliary linear autonomous difference–differential system with commensurable delays

$$\dot{\bar{x}}(t) = D(z)\bar{x}(t) + \bar{b}_1(z)\bar{x}_{n+1}(t),\ t > 0, \tag{2.4}$$

$$\dot{\bar{x}}_{n+1}(t) = v(t),\ t > 0, \tag{2.5}$$

where $\bar{x} = \operatorname{col}[\bar{x}_1, \ldots, \bar{x}_n] \in \mathbb{R}^n$ $(\bar{x}_{n+1} \in \mathbb{R})$ is the solution of system (2.4) and (2.5) and $v(t)$, $t > 0$ is a scalar piecewise continuous control action. There is no need to define concretely the initial condition of system (2.4) and (2.5) for the subsequent considerations. It follows from (2.3) that

$$\operatorname{rank}\begin{bmatrix} \lambda E_n - D(z) & -\bar{b}_1(z) & 0_n \\ 0 & \lambda & 1 \end{bmatrix} = n + 1\ \forall\ \lambda \in \mathbb{C},$$

where $0_n = \operatorname{col}[0, \ldots, 0] \in \mathbb{R}^n$; i.e., for system (2.4), (2.5), complete controllability criterion (1.3) is satisfied. Using the procedure described in [9], we construct a complete damping controller of system (2.4), (2.5). We write this controller as

$$v(t) = \bar{F}_1^1(z)\bar{x}(t) + F_2^1(z)\bar{x}_{n+1}(t) + F_3^1(z)\bar{y}(t), \quad t > 0, \tag{2.6}$$

$$\dot{\bar{y}}(t) = \bar{F}_1^2(z)\bar{x}(t) + F_2^2(z)\bar{x}_{n+1}(t) + F_3^2(z)\bar{y}(t), \quad t > 0, \tag{2.7}$$

where the auxiliary variables $\bar{x}_{n+1} \in \mathbb{R}$ and $\bar{y} = \operatorname{col}[\bar{y}_1, \ldots, \bar{y}_s] \in \mathbb{R}^s$, s is a certain natural number, $\bar{F}_1^1(z) \in \mathbb{R}^{1\times n}[z]$, $F_2^1(z) \in \mathbb{R}^{1\times 1}[z]$, $F_3^1(z) \in \mathbb{R}^{1\times s}[z]$, $\bar{F}_1^2(z) \in \mathbb{R}^{s\times n}[z]$, $F_2^2(z) \in \mathbb{R}^{s\times 1}[z]$, and $F_3^2(z) \in \mathbb{R}^{s\times s}[z]$. In accordance with the construction of the complete damping controller [9], closed-loop system (2.4)–(2.7) is pointwise singular in the directions corresponding to the first $n + 1$ components of its solution, i.e., the components $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$. In other words, there exists a time $t_1 > 0$ such that, regardless of the initial conditions $\bar{x}(t), \bar{x}_{n+1}(t), \bar{y}(t)$ $(t \le 0)$, the following identities hold:

$$\bar{x}_i(t) \equiv 0, \quad t \ge t_1, \quad i = \overline{1, n+1}. \tag{2.8}$$

In relations (1.5)–(1.8) we put $K(z) = \bar{K}(z)R(z) = \operatorname{col}[K_1(z), K_2(z)]$, $K_1(z) \in \mathbb{R}^{r\times n}[z]$, $K_2(z) \in \mathbb{R}^{r_T\times n}[z]$, $F_1^1(z) = \bar{F}_1^1(z)R(z)$, and $F_1^2(z) = \bar{F}_1^2(z)R(z)$; the matrices T, S, and $F_i^j(z)$ for $i = 2, 3$ and $j = 1, 2$ are specified above. Controller (1.5)–(1.8) is constructed.

Let us demonstrate that the constructed controller really ensures equality (1.4) for system (1.1). For this purpose, we consider system (2.1). In this system, we change to new variables by the formula

$$\operatorname{col}[x(t), x_{n+1}(t), y(t)] = \operatorname{diag}[R^{-1}(z), 1, E_s]\operatorname{col}[\bar{x}(t), \bar{x}_{n+1}(t), \bar{y}(t)]. \tag{2.9}$$

As a result, we obtain system (2.4)–(2.7). Transformation (2.9) is nonsingular; hence, (2.8) implies that system (2.1) is pointwise singular in the directions corresponding to the first $n + 1$ variables. Therefore, for any initial function η in (1.2), identity (1.4) holds.

Remark 2. If condition (1.3) holds for the initial system, then in order to construct controller (1.5)–(1.8) we repeat the above reasoning with the matrix $G(z) = 0$. Here, in controller (1.5)–(1.8) it is necessary to put $S = 0$, $T = 0$, and $K_2(z) = 0$; i.e., Eq. (1.6) can be eliminated from the structure of the controller.

3. JUSTIFICATION OF ALGEBRAIC LINK IN THE CONTROLLER'S STRUCTURE

In controller (1.5)–(1.8), we have an algebraic link, which is determined by Eq. (1.6). Therefore, input action (1.5) as a function of argument x on each half-interval $(lh, (l + 1)h]$ $(l = 0, 1, \ldots)$ will change its own structure. If complete-controllability condition (1.3) holds for system (1.1), then (in accordance with Remark 2) an input action considered as a function of the argument x will have the fixed structure. In the general case, it is impossible to construct a controller with an input action of the fixed structure over x

(i.e., a controller of a simpler type) in such a way that the controller ensures (1.4). We justify this by the following result.

Proposition. Suppose that condition (1.3) is violated for system (1.1). If the control $u(t),\ t > 0$ ensures identity (1.4) for system (1.1), then, for $t > t_2 = t_1 + mh$, this control can be represented in the form $u(t) = T\varphi(t) + \mu(t)\ (t > t_2)$, where the function $\varphi(t), t > t_2 - h$ satisfies the difference equation $\varphi(t) = Sz\varphi(t),\ t > t_2$ and the function $\mu(t)\ (t > t_2 - mh)$ satisfies the equation $B(z)\mu(t) = 0,\ t > t_2$ and the initial condition $\mu(t) \equiv 0$ for $t \in (t_2 - mh, t_2]$.

Proof. Assume that condition (1.3) is violated. In this case, if the control $u(t), t > 0$ ensures identity (1.4), then (see [14]) the following equality holds:

$$B(z)u(t) = 0, \quad t > t_2. \tag{3.1}$$

We rewrite (3.1) in the form

$$B_0 u_k(t) + \sum_{i=1}^{m} B_i u_{k-i}(t) = 0, \quad t \in (t_2, t_2 + h], \quad k = 0, 1, \ldots, \tag{3.2}$$

where $u_k(t) = u(t + kh),\ k = -m, -(m-1), \ldots$. For each fixed $t \in (t_2, t_2 + h]$, relation (3.2) is a difference equation of type (1.9). Since (3.2) has the solution $u(t),\ t > 0$, then there exists (see [14]) a piecewise continuous function $f(t)\ (t \in (t_2, t_2 + h])$ such that $u_{-i}(t) = T_i f(t)\ (t \in (t_2, t_2 + h], i = \overline{1, m})$.

Define the functions $\tilde{u}_k(t)\ (t \in (t_2, t_2 + h])$ by the equalities

$$\tilde{u}_{-i}(t) = TS^{m-i} f(t), \quad i = \overline{1, m}, \quad \tilde{u}_k(t) = TS^{m-1}\varphi_k(t), \quad t \in (t_2, t_2 + h], \quad k = 0, 1, \ldots. \tag{3.3}$$

In (3.3), it is taken into account that $T_i = TS^{m-i}\ (i = \overline{1, m})$ and the functions $\varphi_k(t)$ $(t \in (t_2, t_2 + h], k = 0, 1, \ldots)$ are determined by the equation

$$B_0 TS^{m-1}\varphi_k(t) + \sum_{i=1}^{m} B_i TS^{m-i}\varphi_{k-i}(t) = 0, \quad t \in (t_2, t_2 + h], \quad k = 0, 1, \ldots \tag{3.4}$$

and the initial condition $\varphi_{-i}(t) = f(t)\ (t \in (t_2, t_2 + h], i = \overline{1, m})$. Using equality (1.10), straightforwardly verify that the functions $\varphi_k(t) = S\varphi_{k-1}(t)\ (t \in (t_2, t_2 + h], k = 0, 1, \ldots)$ satisfy Eq. (3.4). Hence, functions (3.3) satisfy Eq. (3.2) (for $u_k = \tilde{u}_k$). Define the function φ by the equalities $\varphi(t) = S^{m-1}\varphi_k(t - kh)$, $t \in (t_2 + kh, t_2 + (k+1)h],\ k = -1, 0, \ldots$. Then, the function $\tilde{u}(t) = T\varphi(t)\ (t > t_2)$ satisfies Eq. (3.1) with the initial data $\tilde{u}(t - ih) = T_i f(t),\ t \in (t_2, t_2 + h],\ i = \overline{1, m}$. We put $\mu(t) = u(t) - \tilde{u}(t),\ t > t_2 - mh$. The proposition is proved.

Remark 3. In view of (2.2), $B(z)(T\psi(t) + \mu(t)) = B(z)T\psi(t) = G(z)K_2(z)x(t)$ for $t > t_2 - mh$. Hence, the function μ does no affect (see the lemma) the solution $x(t),\ t > mh$ of closed-loop system (1.1). This determines the form of Eq. (1.5) of the controller.

Remark 4. We can construct a controller that ensures for system (2.1), in addition to identity (1.4), a finite spectrum and asymptotic stability. For this purpose, it is necessary to apply the results of [10, 11] to system (2.8), (2.9). However, a distributed delay will generally appear in the closed-loop system in this case.

4. CONSTRUCTION OF A CONTROLLER FOR A SYSTEM OF NEUTRAL TYPE

In this section, we consider the construction of a controller with a state feedback in the case of a linear autonomous difference-differential system of the neutral type with many commensurable delays

$$\frac{d}{dt}\left(x(t) - \sum_{i=1}^{m} L_i x(t - ih)\right) = \sum_{i=0}^{m} A_i x(t - ih) + \sum_{i=0}^{m} B_i u(t - ih), \quad t > 0, \tag{4.1}$$

where $L_i \in \mathbb{R}^{n \times n}$. All other designations that are used in this section without a clarification have the earlier meaning. For the initial condition of system (4.1), we use set (1.2) with the initial function $\eta \in C\left([-mh, 0], \mathbb{R}^n\right)$ and an arbitrary piecewise continuous function u^0. In accordance with [20], by the solution of system (4.1) we mean a continuous (not necessarily differentiable) function $x(t)$, $t \geq -mh$, that coincides with the function η for $t \in [-mh, 0]$ and satisfies (4.1) almost everywhere.

Definee the polynomial matrix

$$L(z) = \sum_{i=1}^{m} L_i z^i$$

and consider the polynomial matrix

$$G(z) = \sum_{i=0}^{m-1} G_i z^i,$$

where the matrices G_i were defined before (see Section 1). We have the following result [15].

T h e o r e m 2 (the criterion of solution damping). In order for any initial condition (1.2) of system (4.1) a control $u(t)$, $t > 0$ that ensures (1.4) to exist, it is necessary and sufficient that the following equalities simultaneously hold:

$$\text{(i) rank}[\lambda(E_n - L(e^{-\lambda h})) - A(e^{-\lambda h}), B(e^{-\lambda h}), G(e^{-\lambda h})] = n \quad \forall\, \lambda \in \mathbb{C}, \tag{4.2}$$

$$\text{(ii) det}\left[\lambda^m E_n - \sum_{i=1}^{m} \lambda^{m-i} L_i\right] = \lambda^{mn}. \tag{4.3}$$

R e m a r k 5. If we put $G(e^{-\lambda h}) = 0$ in (4.2), then the resultant condition together with condition (4.3) gives the complete-controllability criterion [15] for system (4.1).

Consider the construction of a linear feedback that ensures identity (1.4) for the solution x of system (4.1). We assume that conditions (4.2) and (4.3) hold. It follows from (4.3) that $\det[E_n - L(z)] \equiv 1$. Suppose $\Pi(z)$ is the inverse matrix of the matrix $[E_n - L(z)]$. Introduce the notation $A_\Pi(z) = A(z)\Pi(z)$. Then, in view of (4.2), we have

$$\text{rank}[\lambda E_n - A_\Pi(e^{-\lambda h}), B(e^{-\lambda h}), G(e^{-\lambda h})] = n \ \forall\, \lambda \in \mathbb{C}. \tag{4.4}$$

Consequently, for the linear autonomous difference–differential system (with the solution $\breve{x} \in \mathbb{R}^n$)

$$\dot{\breve{x}}(t) = A_\Pi(z)\breve{x}(t) + B(z)u(t), \quad t > 0, \tag{4.5}$$

there exists a controller (of type (1.5)–(1.8)) that ensures identity $\breve{x}(t) \equiv 0$, $t \geq \hat{t}_1$ for a certain $\hat{t}_1 > 0$ independently of the initial condition of system (4.5). We represent this controller in the form

$$u(t) = K_1(z)\breve{x}(t) + e_1 x_{n+1}(t) + T\psi(t), \quad t > 0, \tag{4.6}$$

$$\psi(t) = Sz\psi(t) + K_2(z)\breve{x}(t), \quad t > 0, \tag{4.7}$$

$$\dot{x}_{n+1}(t) = F_1^1(z)\breve{x}(t) + F_2^1(z)x_{n+1}(t) + F_3^1(z)y(t), \quad t > 0, \tag{4.8}$$

$$\dot{y}(t) = F_1^2(z)\breve{x}(t) + F_2^2(z)x_{n+1}(t) + F_3^2(z)y(t), \quad t > 0, \tag{4.9}$$

where all designations are the same as in (1.5)–(1.8). We substitute (4.6) into Eq. (4.5) to obtain

$$\dot{\breve{x}}(t) = \left(A_\Pi(z) + B(z)K_1(z)\right)\breve{x}(t) + B(z)e_1 x_{n+1}(t) + B(z)T\psi(t), \quad t > 0. \tag{4.10}$$

In Eqs. (4.6)–(4.10), we change the variables: $\breve{x}(t) = [E_n - L(z)]x(t)$. After analyzing the resultant relations, we arrive at the following conclusion: the controller

$$u(t) = K_1(z)[E_n - L(z)]x(t) + e_1 x_{n+1}(t) + T\psi(t), \quad t > 0, \tag{4.11}$$

$$\psi(t) = Sz\psi(t) + K_2(z)[E_n - L(z)]x(t), \quad t > 0, \tag{4.12}$$

$$\dot{x}_{n+1}(t) = F_1^1(z)[E_n - L(z)]x(t) + F_2^1(z)x_{n+1}(t) + F_3^1(z)y(t), \quad t > 0, \tag{4.13}$$

$$\dot{y}(t) = F_1^2(z)[E_n - L(z)]x(t) + F_2^2(z)x_{n+1}(t) + F_3^2(z)y(t), \quad t > 0 \tag{4.14}$$

ensures for the solution of system (4.1) the identity

$$[E_n - L(z)]x(t) \equiv 0, \quad t \geq \hat{t}_1. \tag{4.15}$$

Introduce the notation

$$\hat{x}(t) = \operatorname{col}\left[x(t), \ldots, x(t-(m-1)h)\right], \quad \hat{L} = \begin{bmatrix} L_1 & L_2 & \ldots & L_{m-1} & L_m \\ E_n & 0_{n\times n} & \ldots & 0_{n\times n} & 0_{n\times n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0_{n\times n} & 0_{n\times n} & \ldots & E_n & 0_{n\times n} \end{bmatrix}.$$

We rewrite Eq. (4.15) as

$$\hat{x}(t) = \hat{L}\hat{x}(t-h), \quad t \geq \hat{t}_1. \tag{4.16}$$

Using the Laplace theorem for determinants and (4.3), we obtain the equality $\det[\lambda E_{mn} - \hat{L}] = \lambda^{mn}$, i.e., $\hat{L}$ is a nilpotent matrix. Denote by ξ the index of nilpotency of $\hat{L}$ ($\hat{L}^{\xi} = 0$). It follows from (4.16) and the nilpotency of the matrix $\hat{L}$ that $\hat{x}(t) \equiv 0$ for $t \geq \hat{t}_1 + (\xi - 1)h$. From the last equality, in view of the structure of the matrix $\hat{L}$, we have $x(t) \equiv 0$ for $t \geq t_1$, where $t_1 = \hat{t}_1 + (\xi - m)h.$.

Now we specify a system that is satisfied by the functions x, x_{n+1}, y in the case of system (4.1) closed by controller (4.11)–(4.14). As a result of the change of variables $\breve{x}(t) = \left[E_n - L(z)\right]x(t)$, we obtain closed-loop system (4.5)–(4.9) whose solution satisfies a system of type (2.1). Therefore, in system (2.1), we formally change $x(t)$ to $\left[E_n - L(z)\right]x(t)$ and $\dot{x}(t)$ to $d\left(\left[E_n - L(z)\right]x(t)\right)/dt$. Eventually we obtain a linear autonomous difference–differential system of neutral type with commensurable delays.

5. EXAMPLE

Consider system (4.1) with the matrices ($m = 3$, $n = 2$)

$$L(z) = \begin{bmatrix} 0 & 0 \\ z + z^2 & 0 \end{bmatrix}, \quad A(z) = \begin{bmatrix} 3 - 4z^2 - 2z^3 & 2 + 2z \\ 2 - 2z - \frac{11z^2}{5} - \frac{z^3}{5} & 3 + \frac{z}{5} \end{bmatrix}, \quad B(z) = \begin{bmatrix} 0 & 5z^2 \\ z - z^2 & 4z^2 \end{bmatrix};$$

$h = \ln 2$. For this system, the complete-controllability condition formulated in Remark 5 is violated. We calculate $T = \operatorname{col}[1, 0]$, $S = 1$, and $G(z) = \operatorname{col}[0, z]$. A simple check shows that conditions (4.2) and (4.3) of Theorem 2 are met. We find the matrix

$$\Pi(z) = \begin{bmatrix} 1 & 0 \\ z^2 + z & 1 \end{bmatrix}$$

and construct system (4.5):

$$\dot{\breve{x}}(t) = \begin{bmatrix} 2z + 3 & 2z + 2 \\ z^2 + z + 2 & \frac{z}{5} + 3 \end{bmatrix}\breve{x}(t) + \begin{bmatrix} 0 & 5z^2 \\ z - z^2 & 4z^2 \end{bmatrix}u(t), \quad t > 0. \tag{5.1}$$

Note that complete-controllability condition (1.3) is also violated for system (5.1). Using Section 2, we construct controller (1.5)–(1.8) for system (5.1). Here, the matrix

$$\tilde{B}(z) = \begin{bmatrix} 0 & 5z^2 & 0 \\ z - z^2 & 4z^2 & z \end{bmatrix};$$

hence, we can immediately put $R(z) = E_2$, $\overline{A}(z) = \tilde{A}(z)$, $\overline{B}(z) = \tilde{B}(z)$. Here it is convenient to take $s_1 = 3$. By the procedure described in [19], we obtain

$$\overline{K}(z) = K(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{9}{5} \end{bmatrix}$$

and write out system (2.4), (2.5)

$$\dot{\overline{x}}(t) = \begin{bmatrix} 3 + 2z & 2 + 2z \\ 2 + 2z & 3 + 2z \end{bmatrix}\overline{x}(t) + \begin{bmatrix} 0 \\ z \end{bmatrix}\overline{x}_3(t), \quad t > 0,$$

$\dot{x}_3(t) = v(t), \quad t > 0.$

According to the procedure described in [9], the characteristic polynomial $d(p)$ of closed-loop system (2.4)–(2.7) can be taken in the form $d(p) = p^2(p-1)(p-5)$. We calculate the polynomial matrices $\bar{F}_1^1(z) = F_1^1(z)$, $\bar{F}_1^2(z) = F_1^2(z)$, and $F_i^j(z), i = 2,3, j = 1,2$, which appear in (2.7). As an example, we write

$$F_3^1(z) = \frac{1}{64}z - \frac{35}{64}z^2 + \frac{97}{64}z^3 - \frac{29}{64}z^4 - \frac{49}{32}z^5 + z^6,$$

$$F_3^2(z) = \left(\frac{27905648}{158565} + \frac{1}{62\ln 2}\right)z - \left(\frac{17}{31\ln 2} + \frac{82025584}{158565}\right)z^2 + \left(\frac{63}{62\ln 2} + \frac{8738096}{52855}\right)z^3$$
$$+ \left(\frac{17}{31\ln 2} + \frac{82025584}{158565}\right)z^4 - \left(\frac{32}{31\ln 2} + \frac{54119936}{158565}\right)z^5.$$

We do not write out the remaining matrices $F_1^1(z)$, $F_2^1(z)$, $F_1^2(z)$, $F_2^2(z)$, because they are lengthy. The way for calculating them can be seen in [9] (formulas (18), (20)–(22), (26)). Then, using considerations of Section 2, we obtain controller (4.6)–(4.9):

$$\begin{aligned} u(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \breve{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \psi(t), \quad t > 0, \\ \psi(t) &= \psi(t-h) + \begin{bmatrix} 0 & \frac{9}{5} \end{bmatrix} \breve{x}(t), \quad t > 0, \\ \dot{x}_3(t) &= F_1^1(z)\breve{x}(t) + F_2^1(z)x_3(t) + F_3^1(z)y(t), \quad t > 0, \\ \dot{y}(t) &= F_1^2(z)\breve{x}(t) + F_2^2(z)x_3(t) + F_3^2(z)y(t), \quad t > 0, \end{aligned} \tag{5.2}$$

where $y \in \mathbb{R}$. Now we write out controller (4.11)–(4.14) for the initial system of the example:

$$\begin{aligned} u(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \psi(t), \quad t > 0, \\ \psi(t) &= \psi(t-h) + \begin{bmatrix} -\frac{9}{5}z^2 - \frac{9}{5}z & \frac{9}{5} \end{bmatrix} x(t), \quad t > 0, \\ \dot{x}_3(t) &= F_1^1(z)\begin{bmatrix} 1 & 0 \\ -z^2 - z & 1 \end{bmatrix} x(t) + F_2^1(z)x_3(t) + F_3^1(z)y(t), \quad t > 0, \\ \dot{y}(t) &= F_1^2(z)\begin{bmatrix} 1 & 0 \\ -z^2 - z & 1 \end{bmatrix} x(t) + F_2^2(z)x_3(t) + F_3^2(z)y(t), \quad t > 0. \end{aligned} \tag{5.3}$$

The fulfillment of identity (1.4) for the system under consideration that is closed by constructed controller (5.3) follows (see [9]) from the pointwise singularity of system (2.1) corresponding (see the lemma) to system (5.1), (5.2).

CONCLUSIONS

For linear autonomous difference–differential systems of neutral type with a continuous solution, a procedure for constructing a linear feedback that ensures the damping of the solution to the initial system is proposed. A distinctive feature of the present work is that it presents a way of applying the controller in question to systems that are not complete-controllable.

Of specific interest are systems of the neutral type (4.1) in the case of an absolutely continuous initial function. The existence conditions of an open-loop control that ensures condition (1.4) for a solution of such systems are obtained in [13]. However, it is very difficult to construct a required controller when these conditions are fulfilled. If the characteristic quasi-polynomial of the corresponding homogeneous system is of the delay type, then it follows from Section 4 that the situation becomes opposite. In some cases, the delay type of the characteristic quasi-polynomial can be obtained by using the results of [6].

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