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Bautin Ideal of a Cubic Map

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Abstract—We compute the radical of the ideal generated by the first three focus quantities of maps defined by irreducible branches of a cubic curve on the real plane. It is shown that the ideal is not radical in this case. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Consider a map of the form

$$w = f(z) = -z - \sum_{n=1}^{\infty} a_n z^{n+1}, \qquad z \in \mathbf{R}. \tag{1}$$

Denote by f^p $(p \in \mathbf{N})$ the p^{th} iteration of map (1).

DEFINITION 1. *A singular point $z = 0$ of map (1) is called a center if $\exists\, \epsilon > 0$ such that $\forall\, z : |z| < \epsilon$, the equality $f^2(z) = z$ holds, and a focus otherwise.*

Clearly, if the right-hand side of (1) is a polynomial, then $z = 0$ is a center if and only if $f(z) \equiv -z$.

DEFINITION 2. *A point $z_0 > 0$ is called a limit cycle of map (1) if z_0 is an isolated root of the equation*

$$f^2(z) - z = 0. \tag{2}$$

To investigate bifurcations of limit cycles of map (1), one can find the return (Poincaré) map

$$\mathcal{P}(z) = f^2(z) = z + c_2 z^3 + c_3 z^4 + \cdots \tag{3}$$

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or compute a Lyapunov function [1,2] defined by a formal series as

$$\Phi(z) = z^2 \left(1 + \sum_{k=1}^{\infty} b_k z^k\right), \tag{4}$$

with the property

$$\Phi(f(z)) - \Phi(z) = g_2 z^4 + g_4 z^6 + \cdots + g_{2m} z^{2m+2} + \cdots . \tag{5}$$

We call the coefficients g_{2i} *focus quantities.*

Consider the map $z \to w$, defined implicitly by the equation

$$\Psi(z,w) = w + z + \sum_{i+j=2}^{n} a_{ij} z^i w^j = 0. \tag{6}$$

This equation has an analytic solution of the form (1),

$$w = \tilde{f}(z) = -z + \cdots . \tag{7}$$

DEFINITION 3. *We say that polynomial (6) defines (or has) a center in the origin if the equation $\Psi(z,w) = 0$ has solution (7) such that the map $\tilde{f}$ has a center in the origin, and we say that (6) defines a focus in the origin, if $\tilde{f}$ has a focus.*

Thus, the problem arises as to how we can find in the space of coefficients $\{a_{ij}\}$ the manifold on which the corresponding maps $\tilde{f}$ have a center in the origin and to investigate bifurcations of limit cycles of such maps in a neighborhood of the origin.

The case of the cubic polynomial

$$\Psi(z,w) = z + w + Az^2 + Bzw + Cw^2 + Dz^3 + Ez^2w + Fzw^2 + Gw^3, \tag{8}$$

where $A, B, \dots, G \in \mathbf{C}$, was considered in [1,2].

Denote the real space of coefficients of polynomial (8) by $\mathcal{E}$, δ-neighborhood of $\alpha^* = (A^*, B^*, \dots, G^*) \in \mathcal{E}$ by $U_\delta(\alpha^*)$, and let $\tilde{f}_\alpha$ be map (7) corresponding to a given point $\alpha = (A, B, \dots, G)$ of the parameter space, i.e.,

$$f_\alpha = -z \left(1 + \sum_{k=1}^{\infty} a_k(A, \dots, G) z^k\right). \tag{9}$$

DEFINITION 4. *Let $n_{\alpha,\epsilon}$ be the number of limit cycles of the map f_α in $|z| < \epsilon$. Then we say that a singular point $z = 0$ of the map f_{α^*} has cyclicity k with respect to space $\mathcal{E}$ in the origin if $\exists\, \delta_0, \epsilon_0$ such that for every $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$,*

$$\max_{\alpha \in U_\delta(\alpha^*)} n_{\alpha,\epsilon} = k.$$

In the case when $\Psi(z,w)$ is a quadratic polynomial (i.e., $D = E = F = G = 0$), it was shown in [1] that $\Psi(z,w)$ defines a center in the origin iff one of conditions

(i) $A - B + C = 0$,
(ii) $A - C = 0$

holds, and the cyclicity of the origin for every map $\tilde{f}$ defined by

$$\Psi(z,w)=0$$

equals 0.

In [2], it also was proven that the cyclicity of the singular point $z=0$ of the map $\tilde{f}=-z+\cdots$ defined by polynomial (8) and having a focus in the origin equals 2. Thus, there remains an open problem to investigate bifurcations of small limit cycles from a center.

Let us denote by $\mathcal{I}$ the ideal generated by all focus quantities of the map, defined by polynomial (8), $\mathcal{I}=\langle g_2,g_4,g_6,\dots\rangle$. We call the ideal $\mathcal{I}$ the *Bautin ideal* of polynomial (8). It follows from results of [2] that if the ideal $I_3=\langle g_2,g_4,g_6\rangle$ were radical, then the cyclicity of any center defined by polynomial (8) would be equal to or less than 2.

In the present paper, we show that the ideal I_3 defining the center variety of polynomial (8) is not radical, and therefore, it is impossible to give an estimation for cyclicity of centers defined by polynomial (8) by directly applying Bautin's method [3].

2. COMPUTING OF THE RADICAL OF THE IDEAL I_3.

To find the radical of the ideal $I_3=\langle g_2,g_4,g_6\rangle$, we will use the following proposition proven in [2].

THEOREM 1. *The center variety of polynomial (8) is equal to*

$$\mathbf{V}(S)\cup\mathbf{V}(H)\cup\mathbf{V}(T),$$

where

$S=\langle A-B+C, D-E+F-G\rangle$, $H=\langle A-C, D-G, E-F\rangle$, *and* $T=\langle t_1,t_2,\dots,t_7\rangle$,

with

$$\begin{aligned}
t_1 &= D^2-DF+EG-G^2,\\
t_2 &= -2CDE+BDF+CDF+CEF-AF^2-3BDG+3CDG+4AEG\\
&\quad -2BEG-CEG+BFG-2CFG-3AG^2+3BG^2,\\
t_3 &= -C^2D+C^2E-BCF+C^2F+F^2+B^2G-BCG-C^2G+4DG-4EG-2FG+5G^2,\\
t_4 &= 2AD-BD-CD+CE-AF+AG+BG-2CG,\\
t_5 &= -BCD+C^2D+C^2E-ACF+2DF+2ABG\\
&\quad -ACG-BCG-2DG-4EG+2FG+2G^2,\\
t_6 &= -2B^2D+5BCD-C^2D-4ACE+2BCE-C^2E+8DE+2ABF-ACF-2BCF\\
&\quad +2C^2F-6DF-4EF+2F^2+3ACG-3BCG+2C^2G-10DG+8EG+2FG-8G^2,\\
t_7 &= A^2-AB+BC-C^2-D+E-F+G.
\end{aligned}$$

Focus quantities computing by means of the algorithm from [2] are

$$\begin{aligned}
g_2 &= 2\left(-A^2+AB-BC+C^2+D-E+F-G\right),\\
g_4 &= -2ABD+2ABG+2ACD+2ACE-2ACF-2ACG+2B^2D-2B^2G\\
&\quad -4BCD-2BCE+2BCF+4BCG+2C^2D+2C^2E-2C^2F-2C^2G\\
&\quad +4D^2-4DE+2DF-2DG+2EF-2EG-2F^2+4FG-2G^2,\\
g_6 &= 2ABEG-2ABFG+ACDF-ACDG-ACEF-3ACEG+ACF^2\\
&\quad +2ACFG+ACG^2-2B^2EG+2B^2FG+BCD^2-BCDE-BCDF\\
&\quad +2BCDG+2BCEF+3BCEG-2BCF^2-BCFG-3BCG^2-3C^2D^2\\
&\quad +4C^2DE-C^2DF-C^2DG-C^2E^2-C^2EF-C^2EG+2C^2F^2-2C^2FG\\
&\quad +4C^2G^2+2D^3-2D^2E-2D^2F+4D^2G+4DEF-8DEG-2DF^2+6DFG\\
&\quad -2DG^2+2E^2G-2EF^2+2EFG-2EG^2+2F^3-6F^2G+8FG^2-4G^3.
\end{aligned}$$

We will show that the following statement holds.

THEOREM 2. *The ideal $I_3 = \langle g_2, g_4, g_6\rangle$ generated by the first three focus quantities of map (8) is not radical in $[A, B, \dots, G]$.*

The proposition is a simple corollary of the following lemma.

LEMMA 1. *The ideals S, H, T defined in Theorem 1 are prime.*

PROOF OF THEOREM 2. According to Theorem 1 and Lemma 1,

$$\operatorname{rad}(\mathcal{I}) = \operatorname{rad}(I_3) = S \cap H \cap T.$$

We note that the intersection $V \cap W$ of the ideals $V = \langle v_1, \dots, v_m\rangle$ and $W = \langle w_1, \dots, w_s\rangle$ in $k[x_1, \dots, x_n]$ is equal to the first elimination ideal of the ideal

$$\langle tv_1, \dots, tv_m, (1-t)w_1, \dots, (1-t)w_s\rangle \subset k[t, x_1, \dots, x_n]$$

(see [4, Theorem 11, p. 186]). To compute the first elimination ideal of this ideal, one finds a Groebner basis with respect to a lexicographic order in which t is greater then the x_i and takes the elements of this basis which do not contain the variable t [4, p. 114].

Computing by means of the algorithm, we get the radical of I_3. Then we find that $\operatorname{rad}(I_3)$ and I_3 have different Groebner bases. Therefore, $\operatorname{rad}(I_3) \neq I_3$, i.e., I_3 is not a radical ideal. ∎

Let R be any ring, I be an ideal of R, and $R' = R/I$. For a polynomial $f \in R[x_1, \dots, x_n]$, we denote by $\overline{f}$ the polynomial in $R'[x_1, \dots, x_n]$ obtained by reducing the coefficients of the powers of A, i.e., we have a homomorphism,

$$\begin{aligned} R[x_1, \dots, x_n] &\longrightarrow R'[x_1, \dots, x_n], \\ f &\longmapsto \overline{f}. \end{aligned} \tag{10}$$

We also denote the image of an ideal $K \subset R[x_1, \dots, x_n]$ in $R'[x_1, \dots, x_n]$ by $\overline{K}$.

LEMMA 2. *Let $f_1, \dots, f_l \in R[x]$, $g \in R[x, w]$, $\langle h_1, \dots, h_k\rangle = I \subseteq R$, $R' = R/I, \overline{f_i} \in R'[x], \overline{g} \in R'[x, w]$, then*

$$\langle \overline{f_1}, \dots, \overline{f_l}, \overline{g}\rangle_{R'[x,w]} \cap R'[x] = \overline{\langle f_1, \dots, f_l, g, h_1, \dots, h_k\rangle_{R[x,w]} \cap R[x]}.$$

PROOF. Let $a \in \langle \overline{f_1}, \dots, \overline{f_l}, \overline{g}\rangle_{R'[x,w]} \cap R'[x]$. Then $a = \sum \overline{f_i b_i} + \overline{g}\,\overline{c} \in R'[x]$. Put $X = \sum f_i b_i + \sum gc \in R[x, w]$. There exists $Y \in R[x]$ such that $\overline{Y} = a$. Therefore, $X = Y + \sum h_j R[x, w]$ and then $Y = X - \sum h_j R[x, w] \in R[x]$ and $\overline{Y} = a$. We have $Y \in \langle f_1, \dots, f_l, g, h_1, \dots, h_k\rangle \cap R[x]$ as desired.

Vice versa, let

$$a \in \overline{\langle f_1, \dots, g, h_1, \dots, h_k\rangle_{R[x,w]} \cap R[x]}.$$

Then $\exists\, X = \sum f_i b_i + \sum gc + \sum h_j d_j \in R[x, w]$. Hence, taking into account that $\overline{h_j} = 0$, we get

$$\overline{X} \in R'[x] \cap \langle \overline{f_i}, \overline{g_j}\rangle_{R'[x,w]}. \qquad ∎$$

To complete the proof of Theorem 2, it remains to prove Lemma 1.

We use the method, described in [5], which is taken from the work in [6]. Namely, we use the following statement [5, Corollary 4.4.9, p. 242].

PROPOSITION 1. *An ideal $I \subseteq R[x]$ is prime in $R[x]$ if and only if*

(i) *$I \cap R$ is prime;*
(ii) *$\overline{I}k'[x]$ is prime in $k'[x]$ where $R' = R \cap I$, $k' = R'R'^{-1}$, $- : R \to R'[x]$;*
(iii) *$\overline{I}k'[x] \cap R'[x] = \overline{I}$.*

PROOF OF LEMMA 1. For S and H, the statement of the lemma is obvious. To prove primality of T, we use Proposition 1.

Following Proposition 1, we first need to show that the ideal

$$J_2 = T \cap \mathbf{C}[B, C, \ldots, G]$$

is prime.

Computing, we see that the polynomials $t_1, \ldots, t_7$ form a Groebner basis of the ideal T with respect to the lexicographic order $A > B > C > D > E > F > G$. Therefore, $J_3 = T \cap \mathbf{C}[C, \ldots, G] = \langle t_1 \rangle$ is prime because of irreducibility of t_1 and $J_2 = \langle t_1, t_3 \rangle$. We again apply Proposition 1 to prove that J_2 is prime.

(i) $J_2 = T \cap \mathbf{C}[B, C, \ldots, G] = J_3$ is prime.

(ii) Note that $\bar{J}_2 k'[B] = \langle \bar{t}_3 \rangle$. Let us show that $\langle \bar{t}_3 \rangle$ is irreducible in $k'[B]$.

$$\bar{t}_3 = GB^2 - C(F + G)B + v,$$

where $R' = \mathbf{C}[C, D, E, F, G]/\langle t_1 \rangle$, $k' = \mathbf{C}(C, E, F, G)[\bar{D}]$, with $\bar{D}^2 = F\bar{D} - EG + G^2$ and because of $t_1 = D^2 - FD + EG - G^2$, $v = -C^2\bar{D} + C^2E + C^2F + F^2 - C^2G + 4\bar{D}G - 4EG - 2FG + 5G^2$.

The polynomial $\bar{t}_3$ is irreducible in $k'[B]$ if and only if its discriminant $\mathcal{D} = C^2(F+G)^2 - 4Gv \notin (k')^2$, i.e., it is not a square. If it were a square, then due to unique factorization in $R'[B]$, $\mathcal{D}$ would be reducible in $R'[B]$. However, it is irreducible, because if it were reducible in $R'[B]$, then it would be reducible in $\mathbf{C}[B, C, D, E, F, G]$ also.

(iii) Note that according to Proposition 4.4.4 from [5], if R is an integral domain with k, its quotient field, $I \subset A = R[x_1, \ldots, x_n]$ is a nonzero ideal and $G = \{g_1, \ldots, g_t\}$ is its Groebner basis with respect to some term ordering, then

$$Ik\left[x_1, \ldots, x_n\right] \cap R\left[x_1, \ldots, x_n\right] = IR_s\left[x_1, \ldots, x_n\right] \cap R\left[x_1, \ldots, x_n\right], \tag{11}$$

where $s = \mathrm{lt}(g_1)\mathrm{lt}(g_2) \ldots \mathrm{lt}(g_t)$ and R_s is the localization of R with respect to G. Moreover, if $g \in A$, $g \neq 0$, and y is a new variable, then due to Proposition 4.4.1 from [5],

$$IA_g \cap A = \langle I, yg - 1 \rangle \cap A. \tag{12}$$

We have to show now that $\bar{J}_2 k'[B] \cap R'[B] = \bar{J}_2$. Taking into account Lemma 2 and formulae (11),(12), we get

$$\begin{aligned}
\bar{J}_2 k'[B] \cap R'[B] &= \bar{J}_2 R'_G[B] \cap R'[B] = \langle \bar{t}_3, yG - 1 \rangle \cap R'[B] \\
&= \langle \bar{t}_3, yG - 1 \rangle_{R'[y,B]} \cap R'[B] = \overline{\langle t_3, t_1, yG - 1 \rangle_{R[y,B]} \cap R[B]} \\
&= \overline{\langle t_3, t_1, yG - 1 \rangle_{\mathbf{C}[y,B,C,D,E,F,G]} \cap \mathbf{C}[y, B, C, D, E, F, G]} \\
&= \overline{\langle t_3, t_1, yG - 1 \rangle \cap \mathbf{C}[y, B, C, D, E, F, G]} = \overline{\langle t_3, t_1 \rangle} = \langle \bar{t}_3, \rangle = \bar{J}_2,
\end{aligned}$$

where $y > B > C > D > E > F > G$.

To complete the proof, there remains to consider the ideal T.

(i) The first condition of Proposition 1 has already been proved.

(ii) We have

$$\begin{aligned}
R' &= \frac{\mathbf{C}[B, C, D, E, F, G]}{\langle t_1, t_3 \rangle}, \\
\overline{T} &= TR'[A], \\
k' &= \text{quotient field of } \frac{\mathbf{C}[B, C, D, E, F, G]}{\langle t_1, t_3 \rangle}.
\end{aligned}$$

Note that

$$\overline{T}k'[A] \neq \langle 1 \rangle. \tag{13}$$

Indeed, otherwise, we have

$$\bar{t}_2\bar{\alpha}_2 + \cdots + \bar{t}_7\bar{\alpha}_7 = 1,$$

where $\bar{\alpha}_i \in k'[A]$. Multiplying by a suitable element from R', we get

$$\bar{t}_2\bar{\beta}_2 + \cdots + \bar{t}_7\bar{\beta}_7 = \bar{\beta},$$

where $\bar{\beta}_i \in R'[A]$, $0 \neq \bar{\beta} \in R'$. Hence,

$$t_2\beta_2 + \cdots + t_7\beta_7 + t_1\beta_1 + t_3\beta_3 = \beta,$$

where $\beta_i \in \mathbf{C}[A, B, \ldots, G]$, $\beta \in \mathbf{C}[B, \ldots, G]$. Therefore, $\beta \in \langle t_1, \ldots, t_7 \rangle \cap \mathbf{C}[B, \ldots, G] = \langle t_1, t_3 \rangle$ due to the Groebner basis property. That contradicts $\beta \neq 0$ on R'. It follows from (13) that if among polynomials $\bar{t}_i$, there is $\bar{t}_{i_0}$ of the first degree, then $\overline{T}k'[A] = \langle t_{i_0} \rangle$. Hence, $\overline{T}k'[A] = \langle \bar{t}_2, \bar{t}_4, \bar{t}_5, \bar{t}_6, \bar{t}_7 \rangle = \langle \bar{t}_2 \rangle$ and $\bar{t}_2$ is irreducible, because it is a polynomial of the first degree in $k'[A]$.

(iii) It remains to show that $\overline{T}k'[A] \cap R'[A] = \overline{T}$.

$\overline{T}k'[A] = \langle \overline{t_4} \rangle$. $\overline{t_4}$ has degree 1 in A, and therefore, is irreducible. Indeed, the coefficient of A in $\overline{t_4}$ is equal to $2\overline{D} - \overline{F} + \overline{D}$ and is not zero, because it lies in R' and is polylinear in all variables, but t_1, t_3 are not polylinear.

From (11), (12), and Lemma 2, we get

$$\begin{aligned}\overline{T}k'[A] = \langle \overline{t_4} \rangle k'[A] &= \langle \overline{t_4}, w(\overline{D} - \overline{F} + \overline{D}) - 1 \rangle \cap R'[A] \\ &= \overline{\langle t_4, w(2D - F + D) - 1, t_1, t_3 \rangle \cap R[A]} \\ \subset \overline{\langle t_4, w(2D - F + D) - 1, t_1, t_3, t_4, t_5, t_6, t_7 \rangle \cap R[A]} &= \overline{\langle t_1, \ldots, t_7 \rangle} = \overline{T}.\end{aligned}$$

Therefore, T is prime. ∎

REMARK. Following the approach suggested above, one can also avoid very laborious computation of syzygies in Example 4.4.20 [5].

To conclude, we have shown that the ideal, generated by three first focus quantities of map (8) is not radical, therefore, most probably, the Bautin ideal $\mathcal{I}$ of this map is not radical either. However, it is easily seen that in the case of map (8) with homogeneous perturbations ($A = B = C = 0$) and ($D = E = F = G = 0$), the corresponding ideals are radical. A similar situation takes place for the cyclicity of a singular point of focus or center type in the case of polynomial vector fields. There also, the ideals of quadratic system and the system with homogeneous cubic nonlinearities are radical, however, the ideal of the general cubic system appears to be not radical [7,8].

We also computed the forth focus quantity of map (8) and found that $g_8 \in I_3$. So we believe that Bautin ideal of the map is generated by the three first focus quantities, and therefore, the cyclicity of the map equals two. However, to prove the hypothesis, one needs to develop a method which can be applied in the cases when the Bautin ideal is not radical.

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