ON CLASSIFICATION OF DEGENERATE SINGULAR POINTS OF RICCI FLOWS

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We consider the normalized Ricci flow on generalized Wallach spaces that could be reduced to a system of nonlinear ODEs. As a main result we get the classification of degenerate singular points of the system under consideration in the important partial case $a_i = a_j$, $i, j \in \{1,2,3\}$, $i \neq j$. *In general the problem can also be considered as two-parametric bifurcations of solutions of abstract dynamical systems. Thus the problem under investigation is interesting not only in geometrical sense.*

Key words: Riemannian invariant metric, Einstein metric, generalized Wallach space, Ricci flow, dynamical system, system of nonlinear ordinary differential equations, singular point, degenerate singular point, parametric bifurcations

In the present work we continue investigations started in [1-7]. Consider the autonomous system of nonlinear ODEs obtained in [6]:

$$\frac{dx_1}{dt} = f(x_1,x_2,x_3), \ \frac{dx_2}{dt} = g(x_1,x_2,x_3), \ \frac{dx_3}{dt} = h(x_1,x_2,x_3), \ x_i = x_i(t) > 0, \qquad (1)$$

where $f(x_1,x_2,x_3) = -1 - a_1 x_1\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + x_1 B$,

$$g(x_1,x_2,x_3) = -1 - a_2 x_2\left(\frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3}\right) + x_2 B,$$

$$h(x_1,x_2,x_3) = -1 - a_3 x_3\left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3}\right) + x_3 B,$$

$$B := \left(\frac{1}{a_1x_1} + \frac{1}{a_2x_2} + \frac{1}{a_3x_3} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)^{-1}, \ a_i \in (0,1/2],$$

$i = 1,2,3$

Recall that system (1) arises at investigations of Ricci flows ([8], [9]) on generalized Wallach spaces (see details in [3-5]). As it was proved in [6], system (1) could be equivalently reduced to a system of two differential equations of the type

$$\frac{dx_1}{dt} = \tilde{f}(x_1,x_2), \ \frac{dx_2}{dt} = \tilde{g}(x_1,x_2), \qquad (2)$$

where $\tilde{f}(x_1,x_2) = f(x_1,x_2,\varphi(x_1,x_2))$, $\tilde{g}(x_1,x_2) = g(x_1,x_2,\varphi(x_1,x_2))$, $\varphi(x_1,x_2) = x_1^{-\frac{a_3}{a_1}} x_2^{-\frac{a_3}{a_2}}$.

In Theorems 1-3 of [2] we investigated the case $a_1 = a_2 = b$, $a_3 = c$, important from a geometrical point of view, where $b, c \in (0,1/2]$, and determined all possible values of the parameters b and c ensuring the system (2) degenerate singular points with $x_1 = x_2$ (see [1] for detail). Denote $D := 1 - 4(1 - 2c)(b + c)$. In the present work these investigations are continued. More precisely, we offer a qualitative classification of such singular points. Our main results are contained in Theorems 1-3 (see [6,7]).

Theorem 1. Let $D = 0$. Then for the singular point $(x_1^0, x_2^0) = (2(b+c)q, 2(b+c)q)$ of the system (2) only the following types of singularities are possible:

(a) (x_1^0, x_2^0) is a semi-hyperbolic saddle-node only for $b \in [b_2, 1/4)$, $c = c_1$ or $b \in [b_2, 1/4) \cup (1/4, 1/2]$, $c = c_2$;

(b) (x_1^0, x_2^0) is a linear zero saddle only at $b = 1/4$, $c = 1/4$;

(c) There are no values of b, c such that (x_1^0, x_2^0) could be a nilpotent singular point.

Theorem 2. Let $0 < D < 1$, $\mu = 1 - \sqrt{D}$. Then for the singular point (7) of the system (2) only the following types of singularities are possible:

(a) (x_1^0, x_2^0) is a semi-hyperbolic saddle only at $b \in (0, 1/4)$, $c = c_3$;

(b) There are no values of b, c such that (x_1^0, x_2^0) could be nilpotent or linearly zero singular point.

Theorem 3. Let $D > 0$, $\mu = 1 + \sqrt{D}$. Then for the singular point (7) of the system (2) only the following types of singularities are possible:

(a) (x_1^0, x_2^0) is a semi-hyperbolic saddle only at $b \in (1/4, b_3]$, $c = c_3$;

(b) There are no values of b, c such that (x_1^0, x_2^0) could be nilpotent or linearly zero singular point.

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