Technical University of Białystok, Białystok, Poland

SOLVING THE UNSTEADY HEAT TRANSFER PROBLEM WITH PERIODIC BOUNDARY CONDITION BY THE BOUNDARY INTEGRAL EQUATIONS METHOD[1]

Sorko S.A., Karpovich S.E.

In the paper, a solution of unsteady heat transfer problem with periodic boundary condition by means of boundary integral equations method was presented. Also was presented the results of numerical simulation of heat transfer in two dimensional area with oscillating temperature on the boundary.

1. Introduction

The phenomena of heat transfer witch the periodical changes of temperature on physical boundaries of analysed objects takes place in many engineering mechanisms (engines, compressors), heating and cooling systems and hydraulic networks.

Also the heat transfer equations with conditions of oscillation temperature or heat flux on boundaries takes important role in mathematical description of many engineering, geothermal and biological problems.

2. Equation of the heat conduction

The processes in which the main mechanism of the heat transfer is the mechanism of heat conduction are described by Fourier-Kirchhoff equation.

The differential equation of unsteady heat conduction in homogeneous substance witch constant thermal diffusion coefficient takes the form [1]:

$$\nabla^2 T - \frac{1}{\alpha}\frac{\partial T}{\partial t} = 0 \tag{1}$$

with initial condition (1.a) and boundary conditions (1.b).

$$\mathbf{q} \in \Lambda : T(\mathbf{q}, t=0) = T_{0L}(\mathbf{q})$$

$$\left.\begin{array}{l} \mathbf{p} \in L_1 : T(\mathbf{p}, t) = T_L(\mathbf{p}, t) \\ \mathbf{p} \in L_2 : q(\mathbf{p}, t) = -\lambda \dfrac{\partial T(\mathbf{p}, t)}{\partial n} = q_L(\mathbf{p}, t) \end{array}\right] \qquad (1.a, 1.b)$$

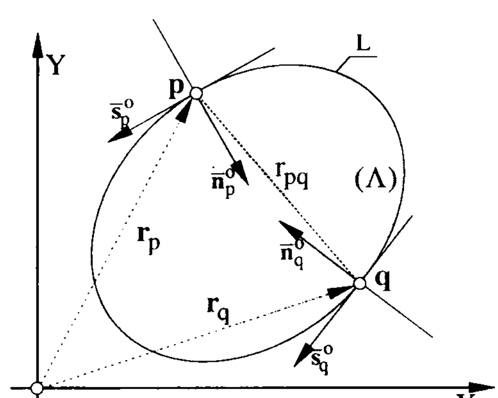

Fig. 1. Sketch for the two dimensional boundary problem analysis of Fourier equation

Particular form of the boundary condition is the condition of periodical changes of the temperature on the boundary:

$$T_L(\mathbf{p}, t) = T(\mathbf{p})\exp(\omega t) \tag{1.c}$$

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The first internal Fourier problem for differential equation (1) with conditions (1.a) and (1.b) in two dimensional area (Λ) has the general solution of the integral form [1]:

$$T(\mathbf{p},t)+\alpha\int_{\tau_0}^{t}\int_{(L_1)}T(\mathbf{q},\tau)E(\mathbf{p},\mathbf{q},t,\tau)dL_q d\tau+\iint_{(\Lambda)}T_{0L}(\mathbf{p})K(\mathbf{p},\mathbf{q},t,\tau)d\Lambda_q=0 \tag{2}$$

where the kernels (fundamental solutions) have the form:

$$\left.\begin{aligned}K(\mathbf{p},\mathbf{q},t,\tau)&=\frac{1}{4\pi\alpha(t-\tau)}\exp\left[-\frac{r(\mathbf{p},\mathbf{q})^2}{4\alpha(t-\tau)}\right]\\ E(\mathbf{p},\mathbf{q},t,\tau)&=\frac{|r(\mathbf{p},\mathbf{q})|\cos(r(\mathbf{p},\mathbf{q}),\bar{\mathbf{n}}_q)}{8\pi\alpha^2(t-\tau)^2}\exp\left[-\frac{r(\mathbf{p},\mathbf{q})^2}{4\alpha(t-\tau)}\right]\end{aligned}\right] \tag{2.a}$$

Function $T(\mathbf{q},\tau)$ satisfies integral equation:

$$-\frac{1}{2}T(\mathbf{p},t)+\alpha\int_{\tau_0}^{t}\int_{(L_1)}q(\mathbf{q},\tau)K(\mathbf{p},\mathbf{q},t,\tau)dL_q d\tau=g(\mathbf{p},t) \tag{2'}$$

where:

$$g(\mathbf{p},t)=T_L(\mathbf{p},t)-\iint_{(\Lambda)}T_{0L}(\mathbf{p})K(\mathbf{p},\mathbf{q},t,\tau)d\Lambda_q \tag{2''}$$

The second internal Fourier problem for differential equation (1) with conditions (1.a) and (1.c) in two dimensional area (Λ) has the general solution of the integral form [1]:

$$T(\mathbf{p},t)+\alpha\int_{\tau_0}^{t}\int_{(L_1)}q(\mathbf{q},\tau)K(\mathbf{p},\mathbf{q},t,\tau)dL_q d\tau+\iint_{(\Lambda)}T_{0L}(\mathbf{p})K(\mathbf{p},\mathbf{q},t,\tau)d\Lambda_q=0 \tag{3}$$

Function $q(\mathbf{q},\tau)$ satisfies integral equation:

$$+\frac{1}{2}q(\mathbf{p},t)+\alpha\int_{\tau_0}^{t}\int_{(L_1)}q(\mathbf{q},\tau)E(\mathbf{p},\mathbf{q},t,\tau)dL_q d\tau=h(\mathbf{p},t) \tag{3'}$$

where:

$$h(\mathbf{p},t)=q_L(\mathbf{p},t)-\iint_{(\Lambda)}T_{0L}(\mathbf{p})E(\mathbf{p},\mathbf{q},t,\tau)d\Lambda_q \tag{3''}$$

3. Solving the problem of the unsteady heat conduction with periodical boundary condition

The unsteady heat conduction in two dimensional object with condition of periodically changed temperature on boundary line is described by the equation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}-\frac{1}{\alpha}\frac{\partial T}{\partial t}=0 \qquad T_L(\mathbf{p},t)=T^*\exp(-\omega t) \tag{4}$$

In this case the temperature may be treated as the function:

$$T=U\exp(-\omega t) \quad T=T(x,y,t)\,;\,U=U(x;y) \tag{5}$$

The space and time derivatives of the temperature are equal respectively:

$$\left.\begin{aligned}
&\frac{\partial T}{\partial x}=\frac{\partial U}{\partial x}\exp(-i\omega t), \frac{\partial^2 T}{\partial x^2}=\frac{\partial^2 U}{\partial x^2}\exp(-i\omega t)\\
&\frac{\partial T}{\partial y}=\frac{\partial U}{\partial y}\exp(-i\omega t), \frac{\partial^2 T}{\partial y^2}=\frac{\partial^2 U}{\partial y^2}\exp(-i\omega t)\\
&\frac{\partial T}{\partial t}=U(-i\omega)\exp(-i\omega t)
\end{aligned}\right] \qquad (5')$$

Introducing relations (5) to equation (4) leads to differential equation for function (U):

$$\nabla^2 U + k^2 U = 0 \quad , \quad k^2 = \frac{i\omega}{\alpha} \qquad (5^*)$$

The integral solution of differential equation has the form [2]:

$$U(\mathbf{p}) = -\int_{(L)} \bar{\mathbf{n}}_q \cdot U(\mathbf{q})K(\mathbf{p},\mathbf{q})dL_q + \int_{(L)} U(\mathbf{p})E(\mathbf{p},\mathbf{q})dL_q \quad ; \quad \mathbf{p},\mathbf{q} \in (\Lambda) \qquad (6)$$

where the kernels $K(\mathbf{r}_p,\mathbf{r}_q)$ and $E(\mathbf{r}_p,\mathbf{r}_q)$ are expressed by spherical Hankel functions of the first kind of order zero and one [3]:

$$\left.\begin{aligned}
&K(\mathbf{p},\mathbf{q}) = \tfrac{i}{4} H_0^{(1)}(kr_{pq})\\
&E(\mathbf{p},\mathbf{q}) = \bar{\mathbf{n}}_q \cdot \nabla K(\mathbf{p},\mathbf{q}) = \tfrac{i}{4} k\, H_1^{(1)}(kr_{pq})
\end{aligned}\right] \qquad (6^*)$$

$$r_{pq} = |\mathbf{p}-\mathbf{q}|$$

On the boundary line (Λ) function $U(\mathbf{q})$; $\mathbf{q} \in (\Lambda)$ satisfies the integral equation:

$$\int_{(L)} \bar{\mathbf{n}}_q \cdot U(\mathbf{q})K(\mathbf{p},\mathbf{q})dL_q = T^* \int_{(L)} E(\mathbf{p},\mathbf{q})\,dL_q - \frac{T^*}{2} \quad ; \quad \mathbf{p},\mathbf{q} \in (L) \qquad (7)$$

4. Numerical solution of integral equation of heat conduction with periodical boundary condition

Discrete solution of integral equation can be obtained approximating the boundary line by the finite set of partial lines. Under condition, that considered equations are satisfied together with boundary conditions on the partial lines (straight or curved elements) (fig. 2).

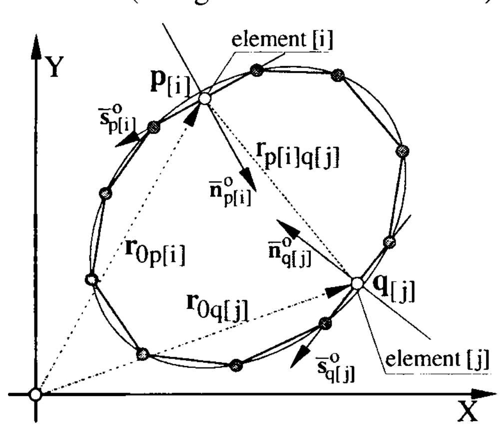

Fig. 2. Discretization of area (Λ)

Integral equation can be expressed as the system of algebraic linear equations for the function $\left(\frac{\partial U(\mathbf{q})}{\partial n}\right)$ at collocation points on each element:

$$\Re\left(\frac{\partial U}{\partial n}\right)_{[i]} \Rightarrow \sum_{j=1}^{I}\left(\frac{\partial U(\mathbf{q}_{[j]})}{\partial n}\right)_{[j]} \Re \int_{(L_{[j]})} K(\mathbf{p}_{[i]},\mathbf{q}_{[j]})dL_q =$$

$$\sum_{j=1}^{J} T^* \left[\left(\Re \int_{(L_{[j]})} E(\mathbf{p}_{[i]},\mathbf{q}_{[j]})dL_q\right) - 0.5\right] \quad ; \quad I = \overline{1,I} \tag{7.a}$$

$$\Im\left(\frac{\partial U}{\partial n}\right)_{[i]} \Rightarrow \sum_{j=1}^{I}\left(\frac{\partial U(\mathbf{q}_{[j]})}{\partial n}\right)_{[j]} \Im \int_{(L_{[j]})} K(\mathbf{p}_{[i]},\mathbf{q}_{[j]})dL_q =$$

$$\sum_{j=1}^{J} T^* \left[\left(\Im \int_{(L_{[j]})} E(\mathbf{p}_{[i]},\mathbf{q}_{[j]})dL_q\right) - 0.5\right] \quad ; \quad I = \overline{1,I} \tag{7.b}$$

In equations (7.a), (7.b) symbols $\Re$ and $\Im$ denotes the real and imaginary part of kernels (6*).

5. Examples

Example (1) – Calculation temperature field in unitary square for the boundary temperature distribution (fig. 3.):

left hand side $T_L = 0^o$ [C] right hand side $T_R = 100^o$ [C] lower side $T_D = 100^o$ [C]

upper side $T_U = 0^o + 100^o \cos(\omega t)$ [C] and $\sqrt{\frac{\omega}{\alpha}} = 1.0$

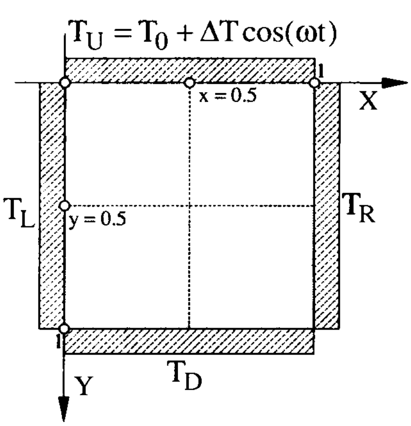

Fig. 3. 2D area

Figures (4.a) and (4.b) illustrated temperature profile respectively at x=0.5 and y=0.5 coordinates for period $(0, 2\pi)$. Figures (5) shows the evolution of the temperature field

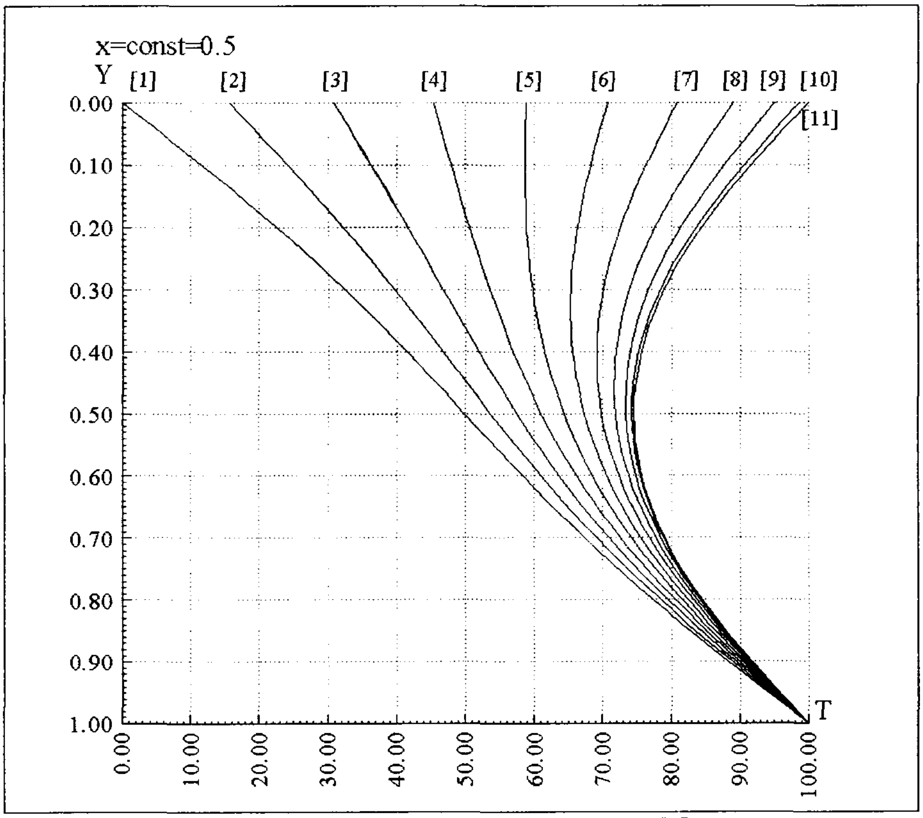

Fig. 4.a. Temperature profile x=const=0.5

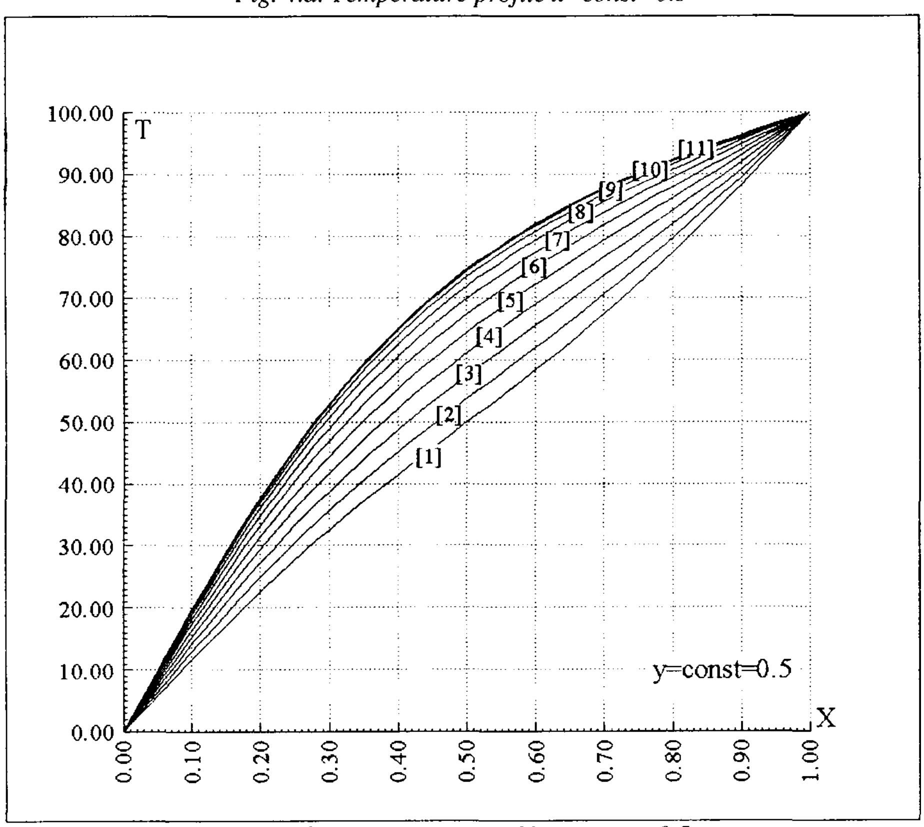

Fig. 4.b. Temperature profile y=const=0.5

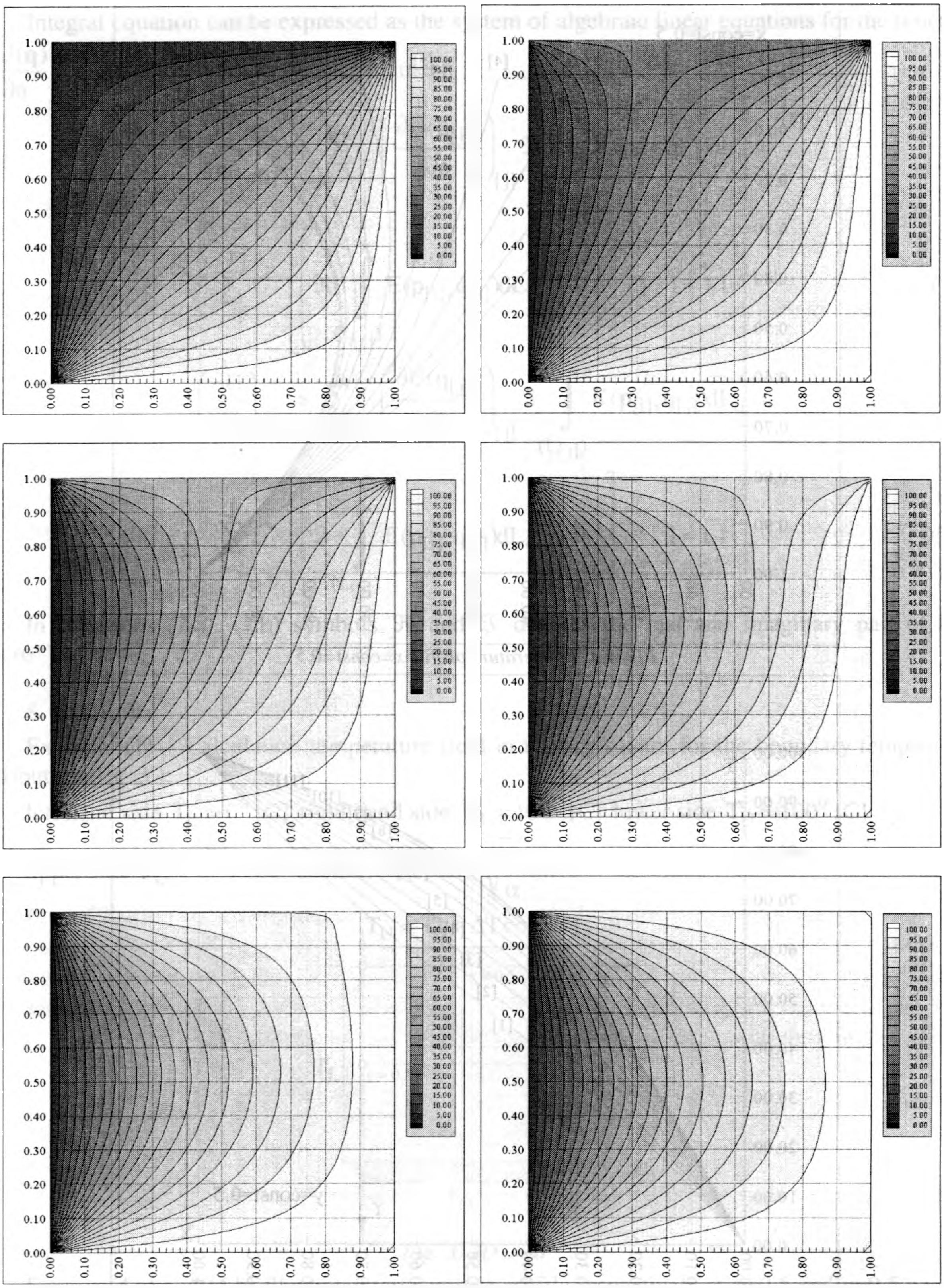

Fig. 5. Temperature field in rectangular area for boundary condition $T_U = T_0 + \Delta T \cos(\omega t)$

Example (2) – Calculation temperature field $\vartheta = f(z,t)$ in semi infinite area with periodical boundary condition $\vartheta = \vartheta^* \cos(\omega t)$ where non-dimensional surplus of the mean temperature of area (ground temperature example in the one year period).

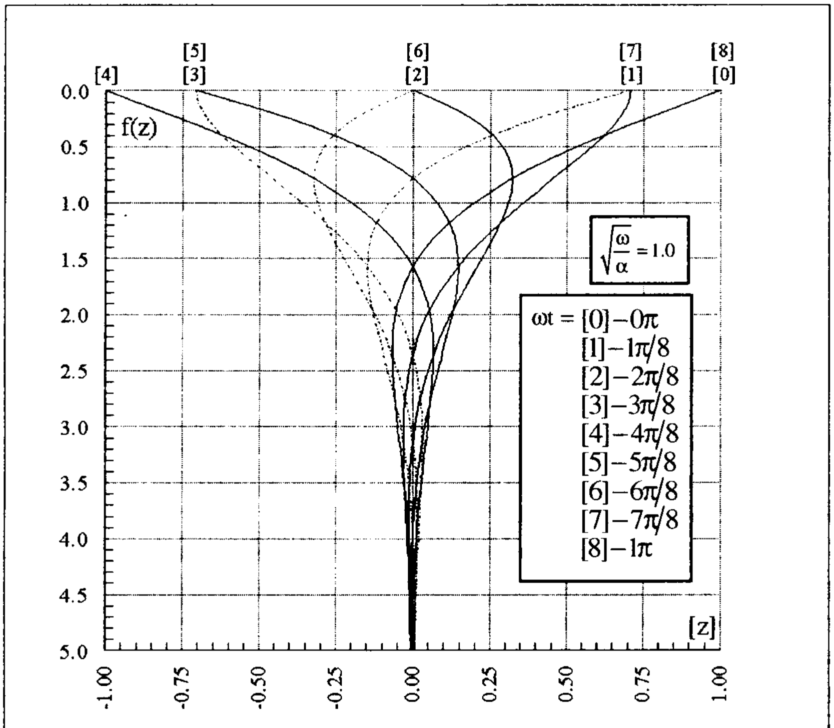

Fig. 6. Temperature field in semi infinite area for boundary condition $\vartheta = \vartheta^{*} \cos(\omega t)$

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