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SOLVING THE OSCILLATING PIPE FLOW PROBLEM BY THE BOUNDARY INTEGRAL EQUATIONS METHOD[1]

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In the work, a solution to a problem of oscillatory flow of viscous incompressible liquid in a rectilinear pipe of circular section, completely filled with the liquid was presented. The way of solving flow generated by pressure pulsation and the pipe oscillation to the flow direction was also discussed. The behaviour of the liquid for the selected spectrum of its frequency and viscosity was further analysed.

1. Introduction

The phenomena of oscillatory flows in hydraulic systems and engineering devices is the important technical problem of designing mechanisms and engineering devices and technical problem of exploitation of hydraulic installations, fluid distribution networks also hydraulic systems of machine admission.

Oscillatory flows in the pipes may be generated by the pressure pulsation or by the oscillatory motion of walls enclosed the flux.

In many cases the phenomena of the oscillatory flows are undesirable because of generation water hammer effects and the cavitation effects. On the other hand the phenomena of oscillatory flows have engineering applications in technology for mixing of immiscible fluids and for aggregation of liquid substances.

Calculation the velocity flow field in conditions of pulsating pressure-driven flow and oscillatory channel motion-driven flow were presented below.

2. Equations of motion of viscous and incompressible fluid

Equations of motion of viscous and incompressible fluid in cylindrical coordinate system (R,θ,Z) (fig. 1.) has the form [1]:

$$\begin{aligned}
&\rho\left(\frac{\partial c_r}{\partial t}+c_r\frac{\partial c_r}{\partial r}+\frac{c_\theta}{r}\frac{\partial c_r}{\partial \theta}-\frac{c_\theta^2}{r}+c_z\frac{\partial c_r}{\partial z}\right)=\rho F_r-\frac{\partial p}{\partial r}+\\
&+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rc_r)\right)+\frac{1}{r^2}\frac{\partial^2 c_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial c_\theta}{\partial \theta}+\frac{\partial^2 c_r}{\partial z^2}\right]\\
&\rho\left(\frac{\partial c_\theta}{\partial t}+c_r\frac{\partial c_\theta}{\partial r}+\frac{c_\theta}{r}\frac{\partial c_\theta}{\partial \theta}+\frac{c_r c_\theta}{r}+c_z\frac{\partial c_\theta}{\partial z}\right)=\rho F_{m\theta}-\frac{1}{r}\frac{\partial p}{\partial \theta}+\\
&+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rc_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 c_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial c_r}{\partial \theta}+\frac{\partial^2 c_\theta}{\partial z^2}\right]\\
&\rho\left(\frac{\partial c_z}{\partial t}+c_r\frac{\partial c_z}{\partial r}+\frac{c_\theta}{r}\frac{\partial c_z}{\partial \theta}+c_z\frac{\partial c_z}{\partial z}\right)=\rho F_{mz}-\frac{\partial p}{\partial z}+\\
&+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 c_z}{\partial \theta^2}+\frac{\partial^2 c_z}{\partial z^2}\right]
\end{aligned} \tag{1}$$

where (ρ) is density of fluid and (μ) is the viscosity.

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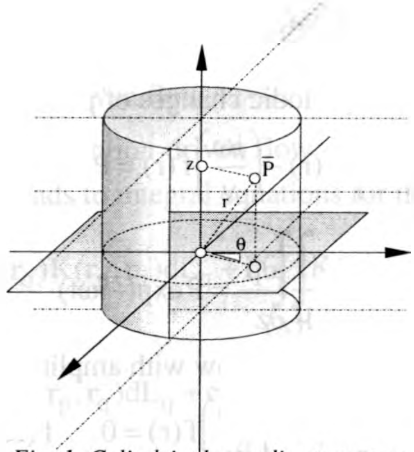

Fig. 1. Cylindrical coordinate system

The flow in straight conduits of circular shape is axisymmetric flow (fig. 2.) and the radial and circumferential components of velocity are equal zero; $c_r = c_\theta = 0$,

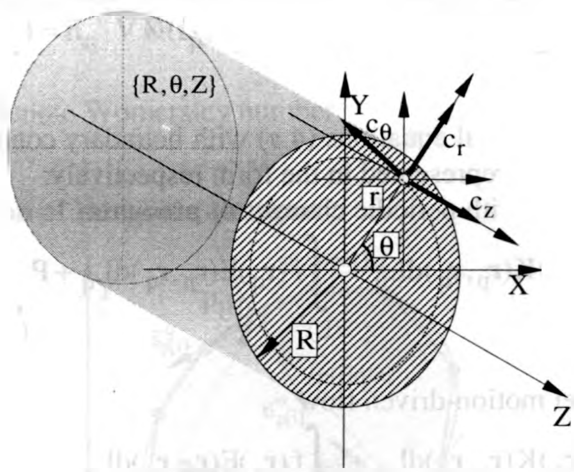

Fig. 2. Axi-symmetrical flow

therefore the motion of the fluid is described by equation for (c_z) component of the velocity:

$$\rho\left(\frac{\partial c_z}{\partial t}+c_z\frac{\partial c_z}{\partial z}\right)=\rho F_{mz}-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 c_z}{\partial \theta^2}+\frac{\partial^2 c_z}{\partial z^2}\right] \tag{2}$$

Assuming that mass forces are negligible $F_{mz}=0$ and $\frac{\partial c_z}{\partial z}=0$; $\frac{1}{r}\frac{\partial c_z}{\partial \theta}=0$ the differential equation describing the flow of viscid and incompressible fluid is obtained:

$$c_z = f(r,t) \quad \frac{\partial c_z}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 c_z}{\partial r^2}+\frac{1}{r}\frac{\partial c_z}{\partial r}\right] \tag{3}$$

The velocity of the periodic flow may be described as the function [2]:

$$c_z(r,t)=f(r)\exp(-i\omega t) \tag{4}$$

The space and time derivatives of velocity are equal respectively:

$$\frac{\partial c_z}{\partial r}=\frac{df(r)}{dr}\exp(-i\omega t) \quad ; \quad \frac{\partial^2 c_z}{\partial r^2}=\frac{d^2 f(r)}{dr^2}\exp(-i\omega t)$$

$$\frac{\partial c_z}{\partial t}=-i\omega f(r)\exp(-i\omega t)$$

so that, the differential equation describing the flow of viscid and incompressible fluid (3) in this case takes the forms:

– for pressure-driven flow with periodic changes of pressure $P\exp(-i\omega t)$:

$$f''(r)+\frac{1}{r}f'(r)-\left(\frac{i\omega}{\nu}\right)f(r)=P \qquad f(R)=0 \tag{4.a}$$

with boundary condition:

$$-\frac{1}{\mu}\frac{\partial p}{\partial z}=P\exp(-i\omega t) \tag{4.a*}$$

– for oscillatory channel motion-driven flow with amplitude (c^*) and frequency (ω):

$$f''(r)+\frac{1}{r}f'(r)-\left(\frac{i\omega}{\nu}\right)f(r)=0 \qquad f(R)=c^* \tag{4.b}$$

with boundary condition:

$$c_z(r,t)=c^*\exp(-i\omega t) \quad ,\frac{1}{\rho}\frac{\partial p}{\partial z}=0 \tag{4.b*}$$

3. Integral equations

The solutions of the differential equations (4.a) with boundary condition (4.a*) and (4.b) with boundary condition (4.b*) can be represented in the form respectively:

– for pressure-driven flow with periodic changes of pressure

$$f(\mathbf{r}_p)=-\int_{(L)}\bar{\mathbf{n}}_q\cdot f(\mathbf{r}_q)K(\mathbf{r}_p,\mathbf{r}_q)dL_q+\int_{(L)}f(\mathbf{r}_q)E(\mathbf{r}_p,\mathbf{r}_q)dL_q+P\iint_{(\Lambda)}K(\mathbf{r}_p,\mathbf{r}_q)d\Lambda_q \quad ; \quad \mathbf{p},\mathbf{q}\in(\Lambda) \tag{5.a}$$

– for oscillatory channel motion-driven flow

$$f(\mathbf{r}_p)=-\int_{(L)}\bar{\mathbf{n}}_q\cdot f(\mathbf{r}_q)K(\mathbf{r}_p,\mathbf{r}_q)dL_q+\int_{(L)}f(\mathbf{r}_q)E(\mathbf{r}_p,\mathbf{r}_q)dL_q \quad ; \quad \mathbf{p},\mathbf{q}\in(\Lambda) \tag{5.b}$$

where the kernels $K(\mathbf{r}_p,\mathbf{r}_q)$ and $E(\mathbf{r}_p,\mathbf{r}_q)$ are expressed by spherical Bessel functions of the first kind of order zero and one:

$$\left.\begin{aligned}K(\mathbf{r}_p,\mathbf{r}_q)&=J_0\left(-\sqrt{\frac{-i\omega}{\nu}}r_{pq}\right)\\ E(\mathbf{r}_p,\mathbf{r}_q)&=\bar{\mathbf{n}}_q\cdot\nabla K(\mathbf{r}_p,\mathbf{r}_q)=-J_1\left(-\sqrt{\frac{-i\omega}{\nu}}r_{pq}\right)\end{aligned}\right],\ r_{pq}=|\mathbf{p}-\mathbf{q}| \tag{5*}$$

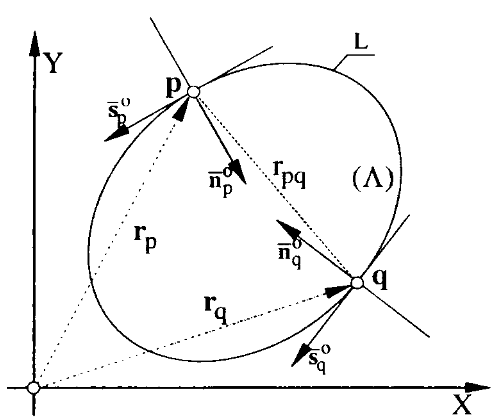

Fig. 3.a. Flow field (Λ)
Sketch for eqs. (5.a), (5.b)

On the boundary of the flow field, there is on the walls enclosing flux the velocity of the flow is equal:

in the case of pulsating pressure-driven flow $\quad f(\mathbf{r}) \equiv f(\mathbf{R}) = 0 \quad$ (eq. 4.a)

in the case of oscillatory channel motion-driven flow $\quad f(\mathbf{r}) \equiv f(\mathbf{R}) = c^* \quad$ (eq. 4.b)

therefore relationships (5.a) , (5.b) leads to integral equations for the function $f(\mathbf{r}_q)\,;\, q \in (L)$:

$$-\int_{(L)} \bar{\mathbf{n}}_q \cdot f(\mathbf{r}_q) K(\mathbf{r}_p, \mathbf{r}_q) dL_q + P \iint_{(\Lambda)} K(\mathbf{r}_p, \mathbf{r}_q) d\Lambda_q = 0 \tag{6.a}$$

$$-\int_{(L)} \bar{\mathbf{n}}_q \cdot f(\mathbf{r}_q) K(\mathbf{r}_p, \mathbf{r}_q) dL_q + c^* \int_{(L)} E(\mathbf{r}_q, \mathbf{r}_q) dL_q - \frac{\pi}{2} c^* = 0 \tag{6.b}$$

where:

$$\left.\begin{array}{l} K(\mathbf{r}_p, \mathbf{r}_q) = J_0(-\sqrt{-i}\,Wo_{pq}) \\ E(\mathbf{r}_p, \mathbf{r}_q) = \bar{\mathbf{n}}_q \cdot \nabla K(\mathbf{r}_p, \mathbf{r}_q) = -J_1(-\sqrt{-i}\,Wo_{pq}) \end{array}\right],\ r_{pq} = |\mathbf{p} - \mathbf{q}| \tag{6*}$$

where $Wo = r\sqrt{\dfrac{\omega}{\nu}} \Rightarrow [-]$ denote Womersley number.

4. Numerical solution of integral equations

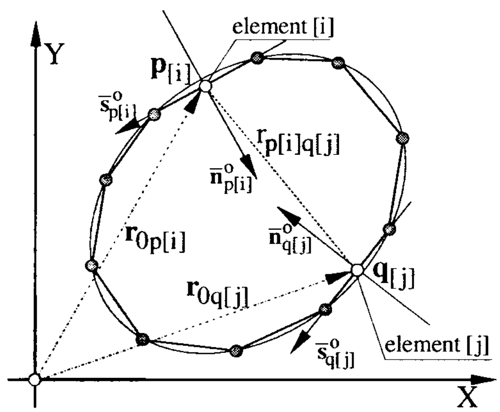

Fig. 3.b. Discretization of area (Λ)

Approximating the integrals over the boundary line (L) with the sum of integrals over the straight line elements $L_{[j]}\,;\, j = \overline{1, J}$ on which the function $\left(\dfrac{\partial f(\mathbf{r}_{q[j]})}{\partial n}\right)_{[j]}$ is constant the discrete formulation of integral equations (6.a) and (6.b) is obtained:

$$\Re \sum_{j=1}^{J} \left(\frac{\partial f(\mathbf{r}_{q[j]})}{\partial n}\right)_{[j]} \int_{(L_{[j]})} K(\mathbf{r}_p, \mathbf{r}_q) dL_{q[j]} = \tag{7.a}$$

$$\Re \sum_{j=1}^{J} P \iint_{(\Lambda_{[j]})} K(\mathbf{r}_{p[i]}, \mathbf{r}_{q[j]}) d\Lambda_{q[j]} = 0 \quad ; \quad i = \overline{1, I}$$

$$\Re \sum_{j=1}^{J} \left(\frac{\partial f(\mathbf{r}_{q[j]})}{\partial n}\right)_{[j]} \int_{(L_{[j]})} K(\mathbf{r}_p, \mathbf{r}_q) dL_{q[j]} = \tag{7.b}$$

$$\Re\sum_{j=1}^{J}\left[c^{*}\int_{(L_{[j]})}K(\mathbf{r}_p,\mathbf{r}_q)\,dL_{q[j]}-\frac{\pi}{2}\right]\quad;\quad i=\overline{1,I}$$

In equations (7.a) and (7.b) sign ($\Re$) denotes real parts of complex kernels (6*).

After determination values of function $\left(\frac{\partial f(\mathbf{r})}{\partial n}\right)_{[i]}\equiv\left(\frac{\partial f(\mathbf{r})}{\partial r}\right)_{[i]}$; $i=\overline{1,I}$ at collocation points (nodal points) on boundary elements velocity is determined in discrete form of formulas (5.a) and (5.b) respectively for case pressure-driven flow and foroscillatory channel motion-driven flow.

5. Examples of oscillatory flows in a circular tube

Example (1) – Calculation the oscillating pressure-driven flow of incompressible and viscous fluid for Womersley number Wo=1.0 and Wo=5.0 (fig. 4.a) and (fig. 4.b).

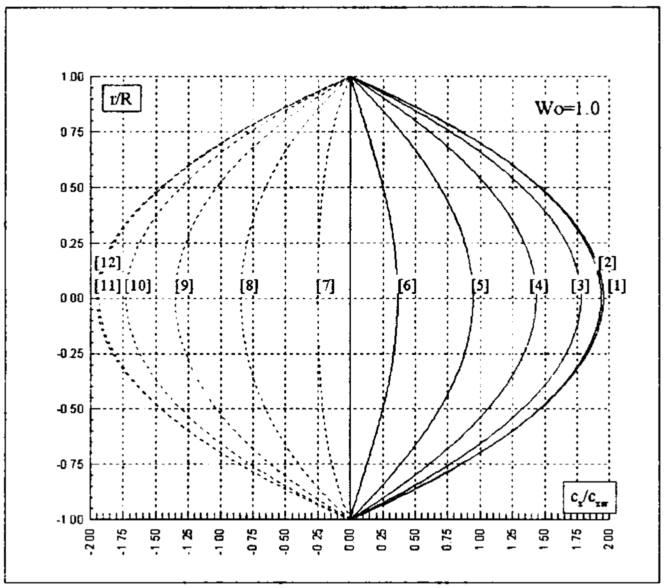

Fig.4.a. Oscillating flow Wo=1.0 (pressure pulsation)

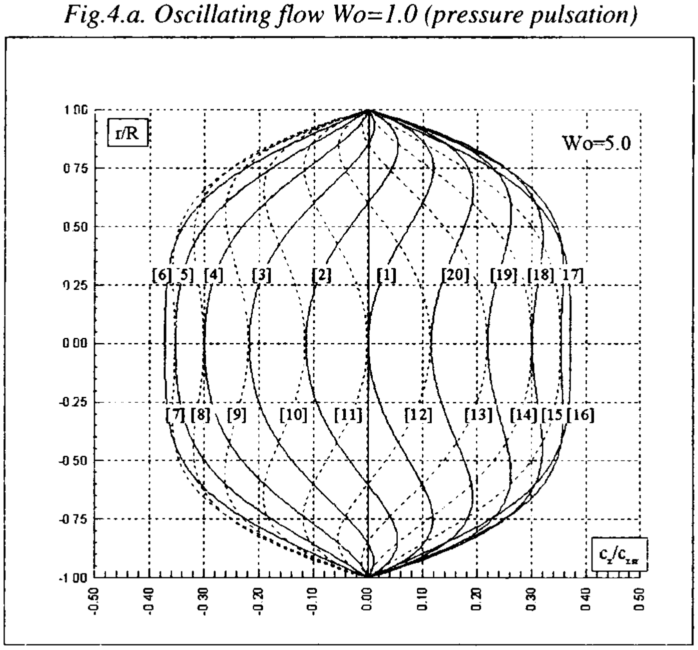

Fig.4.b. Oscillating flow Wo=5.0 (pressure pulsation)

Example (2) – Calculation the oscillatory channel moving-driven flow of incompressible and viscous fluid for Womersley number Wo=1.0 and Wo=5.0 (fig. 5.a) and (fig. 5.b).

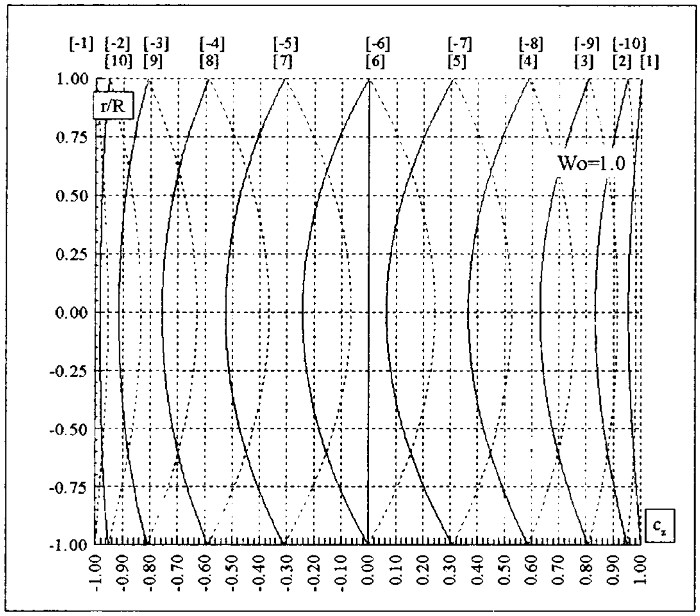

Fig. 5.a. Oscillating flow Wo=1.0 (channel wall oscillation)

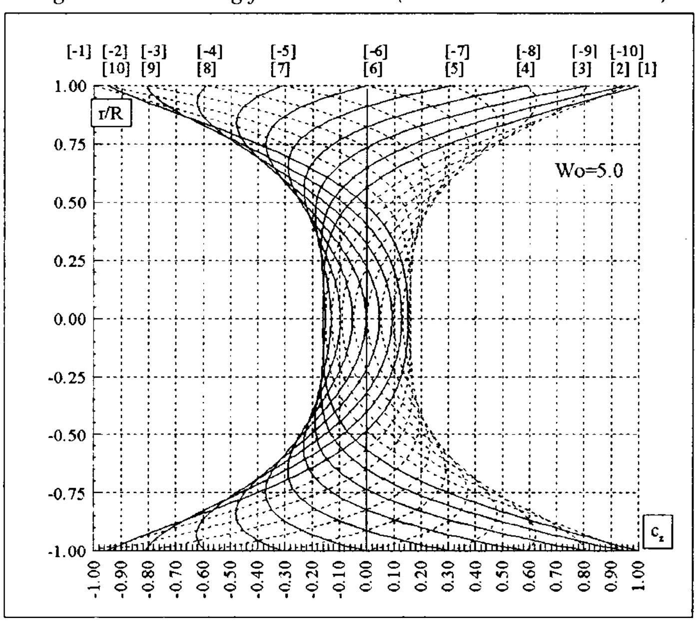

Fig. 5.b. Oscillating flow Wo=5.0 (channel wall oscillation)

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