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THE EFFECTIVE TECHNIQUE OF SOLVING OF REDUCED CHARACTERISTICS EQUATIONS OF RADIATION TRANSFER THEORY

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An integral equation of radiation transfer theory is considered. The method to solution of this equation based on its reduction to an infinity system of linear algebraic equations is presented. The solvability conditions in L_2-space are given and the explicit series solution is constructed.

1. The consideration of different boundary value problems of radiation transfer theory for the case of dispersion media of different configurations [1] require solving of integral equations (IE) of the form:

$$(1 - i\omega\mu)\Phi(\mu\omega) = \frac{\Lambda}{2}\int\limits_{-1}^{1} P_m(\mu,\mu')\Phi(\mu',\omega)d\mu' + g(\mu,\omega), \mu \in [-1,1]. \qquad (1)$$

Here ω and Λ are parameters, $\Phi(\mu,\omega)$ is an unknown, and $g(\mu,\omega)$ is a given function; $\forall m \in N_0 = \{0,1,2,\ldots\}$ $P_m(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi}\chi(\gamma)\cos(m\varphi)d\varphi$, where $\gamma = [(1-\mu^2)(1-\mu'^2)]^{1/2}\cos\varphi + \mu\mu'$ ($\gamma \in [-1,1]$) and $\chi(\gamma)$ is a nonnegative real function, normalized by the condition $\int_{-1}^{1}\chi(\gamma)d\gamma = 2$. The qualitative theory of IE (1) was developed in [2–4], and then in [5] $\forall m \in N_0$ the technique of solving of IE (1) in closed form was developed. Different formal solutions of IE (1) for $m = 0$ were obtained in [3, 6, 7].

In this paper a primary attention is given to formulations and statements substantiating the effective technique of solving of IE (1) yields. This technique allows us a convenient numerical realization which is very important for applications in the field of the radiation transfer theory and dispersion medium optics.

As in [5–7] we use a connection between solutions of IE (1) and an infinite linear set of algebraic equations of truncation procedure, continuous fraction theory and Sturm polynomial systems will be used. Also a part of qualitative and constructive results of [2, 5] will be take into account.

2. Now it is useful to introduce notations. Let C be a set of complex numbers and $\widetilde{S} = [-i\infty, -i] \cup [i, +i\infty]$. Sets of all complex continuous and square integrable functions on $[-1,1]$ are denoted by $C[-1,1]$ and $L_2(0,1)$ respectively. A class of all complex value functions defined on $[-1,1]$, satisfying on $[-1,1]$ the Hölder condition of index α is denoted by $H_\alpha[-1,1]$. A set of all sequences $\{b_s\}$ satisfying the condition $\sum_{s=0}^{+\infty}(2(m+s)+1)(s!/(s+2m)!)|b_s|^2 < +\infty$ will be denoted by $l_2(m)$. An infinite continuous fraction $\alpha_0 + \beta_1(\alpha_1 + \beta_2(\alpha_2 + \ldots)^{-1})^{-1}$ is denoted by $[\alpha_0; \frac{\beta_1}{\alpha_1}; \frac{\beta_2}{\alpha_2}; \ldots]$. A finite continuous fraction is denoted by $[\alpha_0; \frac{\beta_1}{\alpha_1}; \frac{\beta_2}{\alpha_2}; \ldots; \frac{\beta_n}{\alpha_n}]$.

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Besides IE (1) the following infinite system of linear algebraic equations

$$i\omega\varepsilon_{s+1}b_{s+1}(\omega) + i\omega\zeta_s b_{s-1}(\omega) = \varkappa_s b_s(\omega) - v_s(\omega), b_{-1}(\omega) = 0, s \in N_0, \omega \in C \tag{2}$$

will be considered. Here $\{v_s(\omega)\}$ are given and $\{b_s(\omega)\}$ are required functional or numerical sequences; $\{\varepsilon_s\}, \{\zeta_s\}, \{\varkappa_s\}$ are known numerical sequences. Besides (2) the following finite system of linear algebraic equations

$$\begin{cases} i\omega\varepsilon_1 b_1(\omega,n) = \varkappa_0 b_0(\omega,n) - v_0(\omega), \\ i\omega\varepsilon_2 b_2(\omega,n) + i\omega\zeta_1 b_0(\omega,n) = \varkappa_1 b_1(\omega,n) - v_1(\omega), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ i\omega\varepsilon_n b_n(\omega,n) + i\omega\zeta_{n-1} b_{n-2}(\omega,n) = \varkappa_{n-1} b_{n-1}(\omega,n) - v_{n-1}(\omega), \\ i\omega\zeta_n b_{n-1}(\omega,n) = \varkappa_n b_n(\omega,n) - v_n(\omega), n \in N_0 \end{cases} \tag{3}$$

will be used. System (3) is obtained from (2) by using a truncation procedure ($\forall s \in \{n+1, n+2, \dots\}$ $b_s(\omega) \equiv 0$ and $v_s(\omega) \equiv 0$).

3. We formulate statements on qualitative theory of IE (1) and systems (2) and (3).

Theorem 1. *Let $\Lambda \in (0,1), \omega \in C\backslash\widetilde{S}$, a real function $\chi(\mu) \in L_2(-1,1)$ and a function $g(\mu,\omega) \in L_2(-1,1)$ of a variable μ, $\int_{-1}^{1}\chi(\mu)d\mu = 2$, $\forall s,m \in N_0$ $\varepsilon_s = s, \zeta_s = s+2m, \varkappa_s = (2(m+s)+1)(1-\Lambda f_{m+s})$, where $f_s = \frac{1}{2}\int_{-1}^{1}P_s(\mu)\chi(\mu)d\mu$, $P_s(\mu)$ being Legendre polynomials $\forall s,m \in N_0$ $v_s(\omega) = 2((m+s)+1)\Lambda f_{m+s}\int_{-1}^{1}(1-i\omega\mu')^{-1}g(\mu',\omega)P_{m+s}^{m}(\mu')d\mu'$ where $P_{m+s}^{m}(\mu)$ is an associated Legendre polynomial. Then $\forall m \in N_0$ necessary and sufficient conditions for the solvability of IE (1) in the class $L_2(-1,1)$ is the solvability of system (2) in the class $l_2(m)$.*

Theorem 2. *Let $\forall s,m \in N_0$ $\nu_s(m) = \varepsilon_{s+1}\zeta_{s+1}(\varkappa_s\varkappa_{s+1})^{-1}$, $v_s(\omega) \equiv 0$; let the values Λ, ω, ε_s, ζ_s, $\varkappa_s$ satisfy the condition of Theorem 1. Then $\forall m \in N_0$ the necessary and sufficient conditions of the existence of a nonzero solution of system (2) in the class $l_(m)$ is the existence of a nonempty set $\mathcal{R}_m$ of roots of the equation $C_0(\omega^2,m) = \left[1; \frac{\nu_0(m)\omega^2}{1}, \frac{\nu_1(m)\omega^2}{1}, \dots\right] = 0$ with respect to $\omega \in C\backslash\widetilde{S}$. Here $\forall m \in N_0$ there holds $\mathcal{R}_m \subset (-i,i)\backslash\{0\}$.*

These theorems are consequences of Perron's , Pincerle's theorems [8, 9] and of analytical properties of the function $\varphi(z) = \frac{1}{2}(z+z^{-1})$.

Remark 1. For any $m \in N_0$ systems (2) and (3) can have in the class $l_2(m)$ only the unique linearly independent nonzero solution.

Using Theorems 1 and 2 and Fredholm's alternative, the validity of the following statement can be proved.

Theorem 3. *Let the following conditions hold: $\Lambda \in (0,1)$; a real function $\chi(\mu) \in L_2(-1,1)$ and $\int_{-1}^{1}\chi(\mu)d\mu = 2$; $\forall m \in N_0$ the members of the sequences $\{\varepsilon_s\}, \{\zeta_s\}, \{\varkappa_s\}$ are given in Theorem 1; $\forall m \in N_0$ and $\forall\omega \in C\backslash(\widetilde{S} \cup \mathcal{R}_m)$ a sequence $\{(2(m+s)+1)^{-1}v_s(\omega)\} \in l_2(m)$. Then $\forall m \in N_0$ and $\forall\omega \in C\backslash(\widetilde{S} \cup \mathcal{R}_m)$ system (2) has a unique solution $\{b_s(\omega)\}$ in the class $l_2(m)$. In addition, $\forall s \in N_0$ and $\forall\omega \in C\backslash(\widetilde{S} \cup \mathcal{R}_m)$, $b_s(\omega)$ can be presented in the form of a convergence series $b_s(\omega) = \sum_{n=0}^{+\infty}b_{s,n}(\omega)v_n(\omega)$, where $\{b_{s,n}(\omega)\}$ is a unique solution of system (2) in the class $l_2(m)$, when $v_s(\omega) = \delta_{sn}$ $\forall s \in N_0 (n \in N_0)$.*

This theorem is a generalization of Theorem 3 of [5].

Theorem 4. *Suppose that the conditions of Theorem 1 hold. Then $\forall m \in N_0$ and $\forall\omega \in C\backslash(\widetilde{S} \cup \mathcal{R}_m)$ there exists a unique solution of IE (1) in the class $L_2(-1,1)$. This solution can be written as follows:*

$$\Phi(\mu,\omega) = (1-i\omega\mu)^{-1}[g(\mu,\omega)+\widetilde{g}(\mu,\omega)+\frac{\Lambda}{2}\sum_{s=0}^{+\infty}(2(s+m)+1)f_{m+s}\frac{s!}{(s+2m)!}P_{m+s}^{m}(\mu)b_s(\omega)], \mu \in [-1,1]. \tag{4}$$

Here $\widetilde{g}(\mu,\omega) = \frac{\Lambda}{2}\int_{-1}^{1}(1-i\omega\mu')^{-1}P_m(\mu,\mu')g(\mu',\omega)d\mu'$; $\{b_s(\omega)\}$ is a solution of (2) in the class $l_2(m)$. The series in (4) $\forall\omega \in C\backslash(\widetilde{S} \cup \mathcal{R}_m)$ converges in metric $L_2(-1,1)$.

This theorem is a consequence of Theorems 1–3 and IE (1).

It is convenient to introduce in our consideration $\widetilde{D}_{n+1}(\omega, m) = \frac{1}{(n+1)!} D_{n+1}(\omega, m)$, where $D_{n+1}(\omega, m)$ is a determinant of the $(n+1)$-th order of the basic matrix of system (3) ($n \in N$). Denote a set of all zeroes of polynomial $\widetilde{D}_{n+1}(\omega, m)$ by $\mathcal{R}_{m,n_1}$ (n_1 is the order of this polynomial). Then the following theorem is valid.

Theorem 5. *Let a parameter Λ, a function $\chi(\mu)$ and the members of sequences $\{\varepsilon_s\}, \{\zeta_s\}, \{\varkappa_s\}$ satisfy the conditions of Theorem 1. Then $\forall m \in N_0$ and $\forall n \in N$ the following statements are true: 1^0) a polynomial $\widetilde{D}_{n+1}(\omega, m)$ has only zeroes of the first order; a the number of these zeroes is even ($2 \le n_1 \le n+1$); 2^0) $\mathcal{R}_{m,n_1} = \{\pm i k_l(m; n+1) | l = \overline{1, n_2}, n_2 = (n_1/2); k_l(m, n+1) \in \mathcal{R}\backslash\{0\}\}$; 3^0) $\forall r \in N$ there holds the following recurrence formula $\widetilde{D}_{r+1}(\omega, m) = -\widetilde{\varkappa}_r \widetilde{D}_r(\omega, m) + \omega^2 \widetilde{\varepsilon}_r \widetilde{\zeta}_r \cdot \widetilde{D}_{r-1}(\omega, m)$, where $\widetilde{\varepsilon}_r = \varepsilon_r / r$, $\widetilde{\zeta}_r = \zeta_r (r+1)^{-1}$, $\widetilde{\varkappa}_r = (r+1)^{-1} \varkappa_r$, $\widetilde{\varkappa}_0 = \varkappa_0$ and $\widetilde{D}_0(\omega, m) \equiv 1$, $\widetilde{D}_1(\omega, m) = -\varkappa_0$; a set $\mathcal{R}_{m,n_1}$ is equal to the set of all zeroes of a finite continuous fraction $\mathcal{C}_{0,n}(\omega, m) = [1; \frac{\nu_0(m)\omega^2}{1}, \frac{\nu_1(m)\omega^2}{1}, \ldots, \frac{\nu_n(m)\omega^2}{1}]$.*

The proof of this theorem uses Theorem 1 of Chapter 2 of [10], where the properties of a normal Jacobi matrix are given together with symmetrization of the basic matrix of system (3) introducing new unknown quantities $y_0(\omega, n) = \sqrt{\varkappa_0} b_0(\omega, n), y_s(\omega, n) = \sqrt{\varkappa_s} \sqrt{\varepsilon_1 \cdot \ldots \cdot \varepsilon_s (\zeta_1 \cdot \ldots \cdot \zeta_s)^{-1}} b_s(\omega, n)$, where $s \in \{1, \ldots, n\}$.

On the basis of Theorem 2 stated above, Remarks 1 and 2 of [5], Theorem 2 of [3], Pringheim's theorem [11] and Perron's theorem [8] one can prove a number of auxiliary statements which allow to solve IE (1) and systems (2), (3).

Lemma 1. *Let for some $m \in N_0$ the set $\mathcal{R}_m \neq \emptyset, i k_l(m) \in \mathcal{R}_m$ and take place the assumptions of Theorem 2. Then for $\omega = i k_l(m)$ there exists an nonzero solution $\{b_s(i k_l(m))\}$ of system (2) in the class $l_2(m)$. This solution satisfies the relation*

$$\lim_{s \to +\infty} \frac{b_{s+1}(i k_l(m))}{b_s(i k_l(m))} = -k_l(m) \left[1 + \sqrt{1 - k_l^2(m)}\right]^{-1}. \tag{5}$$

Lemma 2. *Let $\Lambda \in (0,1)$ and a function $\chi(\mu)$ satisfies the conditions of Theorem 1. Then $\forall m \in N_0$ the following statements hold:*

1^0) *the set $\mathcal{R}_0$ nonempty;*

2^0) *$(-i k_l(m)) \in \mathcal{R}_m$ if $\mathcal{R}_m \neq \emptyset$ and $i k_l(m) \in \mathcal{R}_m$;*

3^0) *$\forall n \in N_0$ functions $\prod_{r=0}^{n} (\mathcal{C}_r(\omega^2, m))^{-1}$ are analytical in $\mathcal{C}\backslash(\widetilde{S} \cup \mathcal{R}_m)$ and have poles of the first order at any point belonging to $\mathcal{R}_m$.*

Remark 2. If $\Lambda \in (0,1)$, a real function $\chi(\mu) \in H_\alpha[-1,1]$, where $\alpha \in (0,1)$, and $\int_{-1}^{1} \chi(\mu) d\mu = 2$, then the set $\mathcal{R}_0$ is finite.

4. The technique used in [5] to determine explicit solutions of homogeneous and inhomogeneous IE (1) in the class $C[-1,1]$ and system (2) in the class $l_2(m)$ has a natural generalization. It is also applicable to obtain solutions of IE (1) in the class $L_2(-1,1)$ and homogeneous system corresponding to (2) in the class $l_2(m)$.

Further on the basis of the results given in Section 3 statements describing algorithms to obtain analytical and numerical solutions of the equations stated above are formulated.

Theorem 6. *Let the conditions of Theorem 3 hold. Then $\forall m \in N_0$ and $\forall \omega \in \mathcal{C}\backslash(\widetilde{S} \cup \mathcal{R}_m)$ a unique solution $\{b_s(\omega)\}$ of system (2) in the class $l_2(m)$ can be represented in terms of the formulas (12) of [5].*

The proof of this theorem is carried out by using Lemma 2 and Theorem 3.

Corollary 6.1. *Let the assumptions of Theorem 2 hold and the set $\mathcal{R}_m$ be nonempty for some $m \in N_0$. Then $\forall \omega = (i k_l(m)) \in \mathcal{R}_m$ a unique solution of system (2) in the class $l_2(m)$ normalized with the condition $b_0(i k_l(m)) = 1$ can be written in the form*

$$b_s(i k_l(m)) = \begin{cases} 1, s = 0, \\ (-1)^s (k_l(m))^s \prod_{l=1}^{s} \zeta_l (\varkappa_l \mathcal{C}_l(-k_l^2(m), m))^{-1}, s \in N. \end{cases} \tag{6}$$

The validity of this corollary follow from Lemma 2, Theorem 2, 6 and Remark 1.

Remark 3. Let the conditions of Corollary 6 hold. Then for a finite $s \in N_0$ the members of the sequences $\{b_s(ik_l(m))\}$, that is the unique solution of system (2) in the class $l_2(m)$ for $\omega = ik_l(m)$ and $b_0(ik_l(m)) = 1$, can be calculated with the help of backward recursion on the basis of system (2). For $s \gg 1$ a ratio $\dfrac{b_{s+1}(ik_l(m))}{b_s(ik_l(m))}$ should be substituted by the right hand side of formula (5).

Theorem 7. *Let the assumptions of Corollary 6.1 hold. Then* $\forall\omega = (ik_l(m)) \in \mathcal{R}_m$ *the unique solution* $\Phi(\mu, ik_l(m))$ *of IE (1) in the class* $L_2(-1,1)$*, normalized by the condition* $\int_{-1}^{1} P_m^m(\mu)\cdot$ $\cdot\,\Phi(\mu, ik_l(m))d\mu = 1$*, can be written in the form of series*

$$\Phi(\mu, ik_l(m)) = \frac{1}{2}\sum_{s=0}^{+\infty}(2(s+m)+1)\frac{s!}{(s+2m)!}b_s(ik_l(m))P_{m+s}^m(\mu). \tag{7}$$

Here $\forall s \in N_0$ the quantity $b_s(ik_l(m))$ are given by formulas (6), this series being convergence in metric $L_2(-1,1)$.

Corollary 7.1. *The solution* $\Phi(\mu, ik_l(m))$ *of IE (1) in the class* $L_2(-1,1)$ *determined in Theorem 7 and specified by expression (7) belongs to the class* $C[-1,1]$ *if the conditions of Corollary 6.1 hold. In addition, series (7) is convergence absolutely and uniformly at* $[-1,1]$*. At this interval this series is majorized by a number series* $A\sum_{s=0}^{\infty}(m+s)k_l^s(m)$*, where* A *is a finite positive number.*

Theorem 8. *Let the assumptions of Theorem 1 hold. Then* $\forall m \in N_0$ *and* $\forall\omega \in \mathcal{C}\backslash(\widetilde{S} \cup \mathcal{R}_m)$ *the unique solution of IE (1) in the class* $L_2(-1,1)$ *can be represented in the form (4) where the series converges on metric* $L_2(-1,1)$ *and* $\forall s \in N_0$ *the members of the sequence* $\{b_s(ik_l(m))\}$ *should be determined by formulas (12) of [5].*

This theorem follows from Theorem 4 and 6.

To determine of solutions of homogeneous IE, corresponding to IE (1), and infinite homogeneous system (2) ($v_s(\omega) \equiv 0$ $\forall m \in N_0$) it is necessary to demonstrate presence of roots of equation $\mathcal{C}(\omega, m) = 0$ on each interval $[-\beta, \beta]$, where $\beta \in (0,1)$ and then to make their separation and calculations. For this purpose the system of polynomials $\{\widetilde{D}_s(\omega, m)\}$ introduced above can be used.

Theorem 9. *Let the parameter* Λ*, the function* $\chi(\mu)$*, the members of sequences* $\{\varepsilon_s\}, \{\zeta_s\}, \{\varkappa_s\}$ *be satisfied the conditions of Theorem 1 and in addition for some* $m \in N_0$ *the set* $\mathcal{R}_m$ *is nonempty. Then the following statement hold.*

1^0) $\forall(-ik_l(m)) \in \mathcal{R}_m$ $\lim\limits_{n\to+\infty} \widetilde{D}_n(ik_l(m), m) = 0$ *hold.*

2^0) $\forall n \in N$ *the polynomial system* $\widetilde{D}_{n+1}(ik,m), \widetilde{D}_n(ik,m), \ldots, \widetilde{D}_1(ik,m), \widetilde{D}_0(ik,m)$ *is Sturm's polynomial system for the equation* $\widetilde{D}_{n+1}(ik,m) = 0$ *with the respect to* $k \in \mathbf{R}_+$.

3^0) $\forall n \in N$ *the number of roots of the polynomial* $\widetilde{D}_{n+1}(ik,m)$ *at the interval* $[\beta_1, \beta_2], (\beta_1 < \beta_2;$ $\beta_1, \beta_2 \in (0,1))$ *is equal to the change of sign alternations in Sturm's polynomial system in going from* β_1 *to* β_2.

It should be noted that in determining the roots of the polynomial $\widetilde{D}_{n+1}(ik,m)$ at the interval $[\beta_1, \beta_2]$ for $n \gg 1$ one should use the backward recursion, obtained from the recursion formula given in Theorem 5, and the relation $\lim\limits_{n\to+\infty}\dfrac{\widetilde{D}_{n+1}(ik,m)}{\widetilde{D}_n(ik,m)} = -1 + \sqrt{1-k^2}$, where $k \in (0,1)$. The latter relation is satisfied by the minimal solution [9].

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